Common Reflection Surface Analysis

Paraxial approximation for the time of flight of signals traveling in a 2D acoustic medium

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Introduction

I present the derivation in the *midpoint-offset* domain of two formulae approximating, for a shot-receiver pair, the *time of flight* of acoustic signals traveling in a 2D non-homogeneous medium. The first expression assumes a *flat* acquisition surface, while the second, which generalizes the first one, takes into account the *topography* of the ground.

Both formulations are based on the approximated kinematic primary response of a reflector segment located around the incidence point of a *reference normal ray*. The approximation requires the use of three parameters: $\alpha$, $R_{NIP}$ and $R_N$.

The problem is first solved in an *auxiliary* homogeneous medium; the resulting time of flight is then delayed to be accommodated to the final velocity macro-model.

The resulting expression of the traveltime surface $t(x_m, h)$ is known in the litterature as *Common Reflection Surface*. 
Part 1: Case of a flat acquisition surface

These notes were largely inspired by the article presented by G. Höcht, E. de Bazelaire, P. Majer and P. Hubral (1999). I completely derived their work following their steps but trying to be more pedagogic, especially in the section dedicated to the use of an earth auxiliary model for the time of flight correction of seismic signals in real media.
Primary reflections in a 2D non-homogeneous medium

Figure 1: Snell’s law governing rays crossing boundaries in a isotropic medium

\[
\frac{\sin \alpha_i}{v_i} = \frac{\sin \beta_i}{v_{i+1}}
\]
Time of flight for a pair \((S, G)\): analysis in a constant-velocity medium

Figure 2: Kinematic response of the reflector: NIP-ray at point \(R\) with deep \(\alpha\)

**Problem:** determine the line of equal reflection time, \(\overline{SR} + \overline{RG} = v_0 t = 2a\), an *ellipse* tangent to the reflector at point \(R\):

\[
\frac{(x_R - x_m)^2}{a^2} + \frac{y_R^2}{a^2 - h^2} = 1, \quad y_R \tan \alpha = -(x_R - x_m) \left(1 - \frac{h^2}{a^2}\right) \tag{1}
\]

\[x_R = x_0 + R_{NIP} \sin \alpha, \quad y_R = -R_{NIP} \cos \alpha, \quad R_{NIP} = X_0 R \]

where \(2h\) is the acquisition offset, \(x_m\) the midpoint and \(\alpha\) the reflector deep at \(R\).
Solving system (1) in $a^2$ and $x_m$, one finds the following two fundamental relations:

$$t^2(\alpha, h) = \left(\frac{4h}{v_0}\right)^2 + 2 \left[\frac{R_{NIP}(\alpha)}{v_0}\right]^2 \left\{ \sqrt{\left[\frac{h}{\Gamma(\alpha)}\right]^2 + 1 + 1} \right\}, \quad (2)$$

$$x_m(\alpha, h) = x_0(\alpha) + \Gamma(\alpha) \left\{ \sqrt{\left[\frac{h}{\Gamma(\alpha)}\right]^2 + 1 - 1} \right\}, \quad (3)$$

where

$$\Gamma(\alpha) = \frac{R_{NIP}(\alpha)}{2 \sin \alpha}. \quad (4)$$

**Remark:** moving point $R$ along the reflector, see figure (3), changes $\alpha$ near the reference deep $\alpha_0$, providing a family of NIP-rays around the reference one defined by $R_{NIP} = R_{NIP}(\alpha_0)$ and $x_0 = x_0(\alpha_0)$.

**Objective:** compute $t$ as a function of $x_m$ and $h$ around the reference NIP-ray, by eliminating $\alpha$ between (2) and (3).
Approximating $x_0$ and $R_{NIP}$ along a constant-curvature neighborhood of $R$

$$R_N(\alpha) = R_{NIP}(\alpha) + R_R$$

Figure 3: Constant velocity model: small variations around a reference deep $\alpha_0$

$$R_{NIP}(\alpha) = R_{NIP} + R_N \left( \frac{\cos \alpha_0}{\cos \alpha} - 1 \right),$$

$$x_0(\alpha) = x_0 - R_N (\cos \alpha \tan \alpha_0 - \sin \alpha).$$
Paraxial approximation of the kinematic response

The paraxial approximation: development of the time of flight \( t(\alpha, h) \) and the midpoint \( x_m(\alpha, h) \) in power series near the reference NIP-ray at point R.

\[ x_s = x_m(\alpha) - h \quad x_G = x_m(\alpha) + h \]

\[ R_R(\alpha) = R_{NIP}(\alpha) + R_R \]

Figure 4: Kinematic response of the reflector near the reference NIP-ray at point R
Remark 1: around the reference NIP-ray, $R_{NIP}(\alpha)$ and $x_0(\alpha)$ are constrained to satisfy the two auxiliary equations (5) and (6).

Remark 2: seismic reciprocity principle, $t(\alpha, h) = t(\alpha, -h)$, imposes that the Taylor-series expansion of $t$ must contain only even powers of $h$:

$$t(\alpha, h) = \frac{2R_{NIP}}{v_0} + \frac{\partial t(\alpha_0, 0)}{\partial \alpha} (\alpha - \alpha_0) + \frac{\partial^2 t(\alpha_0, 0)}{\partial \alpha^2} (\alpha - \alpha_0)^2 + \frac{\partial^2 t(\alpha_0, 0)}{\partial h^2} h^2 + O(3).$$  

(7)

The same is true for the midpoint Taylor-series expansion, since $x_m(\alpha, h) = x_m(\alpha, -h)$.

Remark 3: the inversion of the midpoint approximation will be used to map

$$(\alpha - \alpha_0, h) \mapsto (x_m - x_0, h),$$

and, finally, represent (7) as a function of the two acquisition parameters $(x_m, h)$.  


Paraxial approximation, first case: \( h = 0 \)

Observe that under this hypothesis equations (2) and (3) take the simple form

\[
t(\alpha) = \frac{2R_{NIP}(\alpha)}{v_0} \quad \text{and} \quad x_m(\alpha) = x_0(\alpha),
\]

from where, via equations (5) and (6), we can compute the two second-order Taylor-series expansions

\[
t(\alpha) = \frac{2R_{NIP}}{v_0} + \frac{2R_N}{v_0} \tan \alpha_0 (\alpha - \alpha_0) + \frac{R_N}{v_0} \left(1 + 2 \tan^2 \alpha_0\right) (\alpha - \alpha_0)^2 + O(3), \tag{8}
\]

\[
x_m(\alpha) = x_0 + \frac{R_N}{\cos \alpha_0} (\alpha - \alpha_0) + \frac{R_N \sin \alpha_0}{\cos^2 \alpha_0} (\alpha - \alpha_0)^2 + O(3), \tag{9}
\]

which provide time of flight and midpoint estimates for ray trajectories near the reference NIP-ray.
**Inversion theorem and Lagrange formula**

**Hypothesis:** let \( x_m(\alpha) \) be analytic at \( \alpha = \alpha_0 \), \( x_m(\alpha_0) = x_0 \), and \( x'_m(\alpha_0) \neq 0 \).

**Conclusion:** then the equation \( x = x_m(\alpha) \) has a *unique* solution \( \alpha = X_m(x) \) such that \( X_m(x_0) = \alpha_0 \). This is a standard theorem in complex analysis.

Since \( x_m \) and \( X_m \) are analytic at \( \alpha = \alpha_0 \) and \( x = x_0 \), they can be expanded in power series,

\[
\begin{align*}
x_m(\alpha) &= p_0 + p_1(\alpha - \alpha_0) + p_2(\alpha - \alpha_0)^2 + \cdots,
\end{align*}
\]
\[
\begin{align*}
X_m(x) &= P_0 + P_1(x - x_0) + P_2(x - x_0)^2 + \cdots,
\end{align*}
\]

that converge in the neighborhood of \( \alpha_0 \) and \( x_0 \). Clearly \( p_0 = x_0 \) and \( P_0 = \alpha_0 \), and the highest coefficients \( P_k \) can be found by equating coefficients in the identity

\[
\alpha - \alpha_0 = \sum_{i=k}^{\infty} P_k \left[ \sum_{j=1}^{\infty} p_j (\alpha - \alpha_0)^j \right]^k.
\]

A general formula for \( P_k \) was found by Lagrange in 1768:

\[
P_k = \frac{1}{k!} \left\{ \frac{d^{k-1}}{d\alpha^{k-1}} \left[ \frac{\alpha - \alpha_0}{x_m(\alpha) - x_0} \right]^k \right\}_{\alpha=\alpha_0}, \text{ for } k = 1, 2, \cdots. (10)
\]
Paraxial approximation, first case: \( h = 0 \) (second part)

Using Lagrange formula (10), the *inversion* of (9) provides the following relation between \( \alpha \) and \( x_m \), valid in the neighborhood of the reference NIP-ray:

\[
\alpha - \alpha_0 = \frac{\cos \alpha_0}{R_N}(x_m - x_0) - \frac{\sin \alpha_0 \cos \alpha_0}{R_N^2}(x_m - x_0)^2 + O(3). \tag{11}
\]

By substituting (11) into equation (8), the zero-offset time of flight for *homogeneous* media can now be represented as a function of the midpoint coordinate \( x_m \):

\[
t(x_m) = 2 \frac{R_{NIP}}{v_0} + 2 \frac{\sin \alpha_0}{v_0}(x_m - x_0) + \frac{\cos^2 \alpha_0}{v_0 R_N}(x_m - x_0)^2 + O(3), \tag{12}
\]

\[
t^2(x_m) = 4 \frac{R_{NIP}^2}{v_0^2} + 8 \frac{R_{NIP} \sin \alpha_0}{v_0^2}(x_m - x_0) +

4 \left( \frac{R_{NIP} \cos^2 \alpha_0 + R_N \sin^2 \alpha_0}{v_0^2 R_N} \right)(x_m - x_0)^2 + O(3). \tag{13}
\]

Equations (12) and (13) are respectively called the *parabolic* and the *hyperbolic* zero-offset expansions.
Paraxial approximation, second case: $x_m = x_0$

Under this hypothesis, equation (3) takes the form
\[
x_0 - x_0(\alpha) = \Gamma(\alpha) \left\{ \sqrt{\left[ \frac{h}{\Gamma(\alpha)} \right]^2 + 1 - 1} \right\} = R_N \left( \cos \alpha \tan \alpha_0 - \sin \alpha \right), \tag{14}
\]
where $x_0(\alpha)$, near the reference NIP-ray, is given by (6).

Observe that after a simple algebraic manipulation of equation (14), we can isolate $h^2$, which, for small angular variations around $\alpha_0$, can be written as
\[
h^2 = -\frac{R_{NIP}R_N}{\sin \alpha_0 \cos \alpha_0} (\alpha - \alpha_0) + O(2),
\]
so that, for $x_m = x_0$, we obtain after inversion
\[
\alpha - \alpha_0 = -\frac{\sin \alpha_0 \cos \alpha_0}{R_{NIP}R_N} h^2 + O(4). \tag{15}
\]
Paraxial approximation, second case: \( x_m = x_0 \) (second part)

Eliminating the square-root between equations (2) and (14), and expanding \( t^2(\alpha, h) \) in Taylor-series around \( \alpha_0 \), we obtain:

\[
t^2(\alpha, h) = 4 \frac{R_{NIP}^2 + h^2}{v_0^2} + 4 \tan \alpha_0 \frac{R_{NIP}R_N}{v_0^2} (\alpha - \alpha_0) + O(2). \quad (16)
\]

After the substitution of (15) into (16), the approximated \textit{hyperbolic} time of flight in for \textit{homogeneous} media can be represented as a function of the acquisition offset \( h \):

\[
t^2(h) = \left( \frac{2R_{NIP}}{v_0} \right)^2 + 4 \frac{\cos^2 \alpha_0}{v_0^2} h^2 + O(4). \quad (17)
\]

From this last expression, approximating the square-root of (17) to the second order in \( h \), we obtain the \textit{parabolic} expansion of the time of flight:

\[
t(h) = 2 \frac{R_{NIP}}{v_0} + \frac{\cos^2 \alpha_0}{v_0 R_{NIP}} h^2 + O(4). \quad (18)
\]
Time of flight in homogeneous media: paraxial approximation

From the Taylor-series expansion (7), collecting (12) and (18), we obtain the parabolic approximation, valid near the reference NIP-ray in constant-velocity media:

\[
t(x_m, h) = \frac{2R_{NIP}}{v_0} + \frac{2 \sin \alpha_0}{v_0} (x_m - x_0) + \frac{\cos^2 \alpha_0}{v_0 R_N} (x_m - x_0)^2 + \frac{\cos^2 \alpha_0}{v_0 R_{NIP}} h^2 + O(3). \tag{19}
\]

Similarly, using (13) and (17), we derive the hyperbolic approximation of the time of flight:

\[
t^2(x_m, h) = \left( \frac{2R_{NIP}}{v_0} \right)^2 + \frac{8R_{NIP} \sin \alpha_0}{v_0^2} (x_m - x_0) + 4 \frac{R_{NIP} \cos^2 \alpha_0 + R_N \sin^2 \alpha_0}{v_0^2 R_N} (x_m - x_0)^2 + \frac{4 \cos^2 \alpha_0}{v_0^2} h^2 + O(3). \tag{20}
\]

The objective now is to adapt equations (19) and (20) to the more general case of a laterally inhomogeneous medium.
Non-homogeneous medium

Figure 5: Kinematic of a laterally inhomogeneous medium.
Figure 6: Kinematic of quasi similar subsurface models: the image point $P_{NIP}^*$ of $P_{NIP}$ is determined by the direction and the radius of curvature of the true NIP-wavefront.
Kinematics of quasi similar subsurface models

Let us find a relation between the time of flight \( t(x_m, h) \) of a signal traveling in a non-homogeneous medium with a constant near-surface velocity \( v_0 \), see figure (5), and the traveltime \( t_A(x_m, h) \), provided by equation (19) and (20), of a signal propagating in an auxiliary homogeneous medium where \( v = v_0 \), see figure(6).

Both wavefronts initiated at \( P_{\text{NIP}} \) and \( P_{\text{NIP}}^* \) are locally identical when they emerge at the surface but they are delayed because of the different action of the two media.

![Figure 7: NIP-wavefronts at three different instants of time: emerging at the source position, at \( x_0(\alpha) \) and at the receiver position, see figure(6).](image)

Figure 7: NIP-wavefronts at three different instants of time: emerging at the source position, at \( x_0(\alpha) \) and at the receiver position, see figure(6).
Time delay between real and auxiliary media (I)

The kinematic illustrated in figure (7) is the same for both media, the real and the auxiliary one. In particular, making the two following assumptions,

(i) the emerging NIP-wave in locally circular and
(ii) it propagates with a constant velocity $v_0$ near the ground surface,

we have that:

$$P_0P_1 = \left(\frac{t'_0}{2} - t_1\right) v_0, \quad P_2P_0 = \left(t_2 - \frac{t'_0}{2}\right) v_0,$$

where for both media $t_1 + t_2$ is the source-receiver time of flight while $t'_0$ is the NIP-ray traveltime.

Hence the quantity $P_2P_0 - P_0P_1 = (t_1 + t_2 - t'_0)v_0$ is preserved in both media, so that we may write:

$$\frac{P_2P_0 - P_0P_1}{v_0} = t(x_m, h) - t_0(\alpha)$$

$$= t_A(x_m, h) - \frac{2R_{NIP}(\alpha)}{v_0}. \quad (21)$$
Time delay between real and auxiliary media (II)

Figure 8: NIP-waves locally circular and concentric for both real and auxiliary media.

Assuming that both NIP-waves, when emerging at $x_0$ and at $x_0(\alpha)$, are locally circular and concentric, see figure (8), the time difference between these two events can be computed monitoring the propagation of an N-wave in the auxiliary subsurface model.
Time delay between real and auxiliary media (III)

As illustrated in figure (8), we find the following time difference, valid for small angular deviations around $\alpha_0$:

\[
t_0(\alpha) - t_0 = \frac{2}{v_0} [R_N(\alpha) - R_N]
\]

\[
= \frac{2}{v_0} [R_{NIP}(\alpha) - R_{NIP}] 
\tag{22}
\]

Finally, combining equations (21) and (22), we obtain the fundamental time correction to $t_A$,

\[
t_A(x_m, h) = t(x_m, h) - \left( t_0 - \frac{2R_{NIP}}{v_0} \right).
\tag{23}
\]

This last relation allows the estimate of the time of flight for a signal traveling in a real medium.
Paraxial approximation of the time of flight in non-homogeneous media

With the help of the auxiliary model, using the time correction (23), equations (19) and (20) provide an estimate of the time of flight in non-homogeneous media where the sound velocity $v_0$ near the ground surface is constant.

Consequently, the parabolic approximation takes the form

$$t(x_m, h) = t_0 + \frac{2\sin\alpha_0}{v_0}(x_m - x_0) + \frac{\cos^2\alpha_0}{v_0R_N}(x_m - x_0)^2 + \frac{\cos^2\alpha_0}{v_0R_{NIP}} h^2 + O(3), \quad (24)$$

while the hyperbolic one becomes

$$\left[ t(x_m, h) - \left( t_0 - \frac{2R_{NIP}}{v_0} \right) \right]^2 = \left( \frac{2R_{NIP}}{v_0} \right)^2 + \frac{8R_{NIP}\sin\alpha_0}{v_0^2}(x_m - x_0) +$$

$$\quad + \frac{4R_{NIP}\cos^2\alpha_0 + R_N\sin^2\alpha_0}{v_0^2R_N}(x_m - x_0)^2 + \frac{4\cos^2\alpha_0}{v_0^2} h^2 + O(3). \quad (25)$$

Both expansions depend, for each point $(x_0, t_0)$ of the zero-offset section, on three parameters, $(\alpha, R_{NIP}, R_N)$, whose values are those for which the traveltime surface fits best the reflection events. The use of (25) is highly recommended for its accuracy.
Part 2: Case of an acquisition surface with topography

The results presented in these notes were inspired by the article presented by P. Chira, M. Tygel, Y. Zhang and P. Hubral. I derived their work in a completely different way, following step by step the approach presented in the first part of this document, then using the concept of auxiliary model.
Acquisition with topography in a homogeneous medium

I present the paraxial approximation for the time of flight of signals traveling in a 2D media with topography and derive formulas for the homogeneous case.

The case of a non-homogeneous medium can be solved by delaying, exactly as we did before, the time of flight corresponding to the auxiliary subsurface model.

Figure 9: Acquisition with topography, kinematic response of point R with deep $\alpha_0$: NIP-ray emerging at point O.
Curved measurement surface

Figure 10: The local reference system is tangent to the surface.

We adopt a local coordinate system, tangent to the surface and centered at $x_0$ (the emerging point of the reference NIP-ray), figures (9). At the origin of the axis, the surface has a curvature $\kappa = 2K$, so that, the topography, see figure (10), may be locally approximated by the parabola

$$y(x) = -Kx^2.$$  \hspace{1cm} (26)
The local reference system

Under the assumption of regularity of the topography, equation (26), in the local reference system the two midpoint coordinates take the form:

\[ x_m = -\frac{\tan \beta}{2K}, \quad y_m = -K \left( x_m^2 + h^2 \cos^2 \beta \right). \] (27)

Eliminating \( \beta \) between equations (27), we see that the time of flight \( t \) is a function of only two variables, \( x_m \) and the acquisition offset \( h \).

The reciprocity principle imposes that the Taylor-series expansion of \( t \) must contain only even powers of \( h \).

This means that, to construct the second-order expansion of \( t \), by analogy with equation (7), we can again discriminate the two acquisition configurations

- \( h = 0 \) with the midpoint coordinate \( y_m = -Kx_m^2 \),
- \( x_m = 0 \) with the midpoint coordinate \( y_m = -Kh^2 \).

Remark: in the local reference system, the second case is equivalent to set \( \alpha = \alpha_0 \).
Computing $x_0(\alpha)$ and $R_{NIP}(\alpha)$ for a NIP-ray reaching the curved surface (I)

Figure 11: NIP-ray leaving the reflector of equation $y_R = g(x_R)$. 
Useful trigonometric relations

With the help of figure (12), we see that

\[ \tan \alpha = \frac{x_C - x_R}{g(x_R) - y_C}, \]  

(28)

where

\[ x_R = x_0 + R_{NIP}(\alpha) \sin \alpha, \quad y_R = -\left[ Kx_0^2 + R_{NIP}(\alpha) \cos \alpha \right], \]  

(29)

and

\[ x_C = R_N \sin \alpha_0, \quad y_C = -R_N \cos \alpha_0. \]  

(30)

Remark that with this notation \( x_0 = x_0(\alpha) \) and \( x_R = x_R(\alpha) \).

In addition, the radius of curvature of the reflector being constant around R, we can write \( R_R = R_N - R_{NIP} = R_N(\alpha) - R_{NIP}(\alpha) \), so that, for deep angles near \( \alpha_0 \), we have

\[ \frac{x_R}{\sin \alpha} = R_N \left( \frac{\sin \alpha_0}{\sin \alpha} - 1 \right) + R_{NIP}. \]  

(31)
Computing $x_0(\alpha)$ for a NIP-ray reaching the curved surface

We want to determine $x_0 = x_0(\alpha)$, the impact point at the surface of a NIP-ray traveling near the reference one, figure (12):

$$\begin{bmatrix} x_0 \\ -Kx_0^2 \end{bmatrix} = \begin{bmatrix} x_R \\ g(x_R) \end{bmatrix} + \lambda \begin{bmatrix} -\tan \alpha \\ 1 \end{bmatrix}. \quad (32)$$

Eliminating $\lambda$ in (32), with the help of equations (28), (29) and (30), we finally find:

$$x_0(\alpha) = \frac{1}{2K \tan \alpha} \left\{ 1 - \sqrt{1 + 4KR_N \tan \alpha (\tan \alpha \cos \alpha_0 - \sin \alpha_0)} \right\}. \quad (33)$$

Observe that, first, $x_0(\alpha_0) = 0$ and, second, taking the limit $K \to 0$ we recover the planar case, equation (6).
Computing \( R_{NIP}(\alpha) \) for a NIP-ray reaching the curved surface

We start with equation (32) and observe that \( OR = |\lambda|/\cos \alpha \) and \( \lambda = -[Kx_0^2 + g(x_R)] \). Thus we can write:

\[
R_{NIP}(\alpha) = \frac{Kx_0^2 + g(x_R)}{\cos \alpha}.
\] (34)

Using (31) and (33), equation (34) takes the form

\[
R_{NIP}(\alpha) = R_{NIP} + R_N \left( \frac{\sin \alpha_0}{\sin \alpha} - 1 \right) - \frac{\cos \alpha}{2K \sin^2 \alpha} \left\{ 1 - \sqrt{1 + 4KR_N \tan \alpha (\tan \alpha \cos \alpha_0 - \sin \alpha_0)} \right\}.
\] (35)

Observe that, first, \( R_{NIP}(\alpha_0) = R_{NIP} \) and, second, taking the limit \( K \to 0 \) we recover the planar case, equation (5).
First acquisition configuration: \( h = 0 \)

![Diagram of zero-offset acquisition on a curved surface.](image)

Figure 12: Zero-offset acquisition on a curved surface.

The line of equal reflection time, tangent to the reflector, is the circle centered in \( x_m \) with radius \( R_{NIP} \). In this source-receiver configuration, we have for any deep \( \alpha \)

\[
x_m(\alpha) = x_0(\alpha), \quad \frac{v_0 t}{2} = R_{NIP}(\alpha).
\]  

(36)

To set up the paraxial approximation, let us perturb \( x_0(\alpha) \), equation (33), and \( R_{NIP}(\alpha) \), equation (35), around the reference angle \( \alpha_0 \) corresponding to \( x_m(\alpha_0) = x_0 = 0 \).
Paraxial approximation for $h = 0$ (first part)

The second-order Taylor series expansion of equations (36) take the form

$$
t(\alpha) = \frac{2R_{NIP}}{v_0} + \frac{2R_N}{v_0} \tan \alpha_0 (\alpha - \alpha_0) +$$

$$+ \frac{R_N}{v_0} \left(1 + 2 \tan^2 \alpha_0 - \frac{2KR_N}{\cos^3 \alpha_0}\right)(\alpha - \alpha_0)^2 + O(3), \quad (37)$$

$$x_m(\alpha) = \frac{R_N}{\cos \alpha_0} (\alpha - \alpha_0) +$$

$$+ \frac{R_N \sin \alpha_0}{\cos^2 \alpha_0} \left(1 - \frac{KR_N}{\cos \alpha_0}\right)(\alpha - \alpha_0)^2 + O(3), \quad (38)$$

generalizing equations (8) and (9) for the case of a surface with a local curvature $\kappa = 2K$ at $x_0 = 0$. 
Paraxial approximation for $h = 0$ (second part)

Using Lagrange formula (10), the inversion of (38) provides the following relation between $\alpha$ and $x_m$, valid in the neighborhood of the reference NIP-ray:

$$\alpha - \alpha_0 = \frac{\cos \alpha_0}{R_N} x_m + \frac{\sin \alpha_0}{R_N^2} (KR_N - \cos \alpha_0) x_m^2 + O(3). \quad (39)$$

After the substitution of (39) into (37), the zero-offset time of flight for homogeneous media can now be represented as a function of the midpoint coordinate $x_m$:

$$t(x_m) = 2 \frac{R_{NIP}}{v_0} + 2 \frac{\sin \alpha_0}{v_0} x_m + \frac{\cos \alpha_0}{v_0 R_N} (\cos \alpha_0 - 4KR_N) x_m^2 + O(3), \quad (40)$$

$$t^2(x_m) = 4 \frac{R_{NIP}^2}{v_0^2} + 8 \frac{R_{NIP} \sin \alpha_0}{v_0^2} x_m +$$

$$\frac{4}{v_0^2} \left( \frac{R_{NIP} \cos^2 \alpha_0 + R_N \sin^2 \alpha_0}{R_N} - 2KR_{NIP} \cos \alpha_0 \right) x_m^2 + O(3). \quad (41)$$

Equations (40) and (41) are respectively the parabolic and the hyperbolic zero-offset expansions for the case with a local curvature $\kappa = 2K$ at $x_0 = 0$. 
Second acquisition configuration: $x_m = 0$ and $y_m = -Kh^2$ (first part)

![Diagram](image)

Figure 13: Acquisition with offset $h$ on a curved surface.

The line of equal reflection time, tangent to the reflector at point $R$, is the ellipse centered at the midpoint $M$ with foci on source and receiver positions:

$$\frac{x_R^2}{v_0^2 t^2} + \frac{(y_R + Kh^2)^2}{v_0^2 t^2 - 4h^2} = \frac{1}{4}, \quad \text{where} \quad \frac{x_R}{y_R + Kh^2} = -\frac{v_0^2 t^2}{v_0^2 t^2 - 4h^2} \tan \alpha, \quad (42)$$

The coordinates $(x_R, y_R)$ of the reflecting point $R$ are given by (29).
Second acquisition configuration: \( x_m = 0 \) and \( y_m = -Kh^2 \) (second part)

Using (29), the system of equations (42) provides the following solution:

\[
\left( \frac{v_0 t}{2} \right)^2 = h^2 + \frac{\Delta y_R^2}{\cos^2 \alpha} - \Delta y_R \tan \alpha \left[ x_0(\alpha) - K(x_0^2 - h^2) \right], \tag{43}
\]

\[
x_0(\alpha) = K \left[ x_0^2(\alpha) - h^2 \right] \tan \alpha - \Gamma(\alpha) \left[ \sqrt{\frac{h^2}{\Gamma^2(\alpha)} + 1} - 1 \right], \tag{44}
\]

where

\[
\Delta y_R = K \left[ x_0^2(\alpha) - h^2 \right] - R_{NIP} \cos \alpha, \quad \Gamma(\alpha) = \frac{\Delta y_R}{2 \sin \alpha \cos \alpha}.
\]

With the help of expression (33), eliminating \( x_0 \) in both equations (43) and (44), we are ready to construct the paraxial approximation of \( t = t(\alpha) \) and \( h = h(\alpha) \), developing in Taylor series these two functions around \( \alpha_0 \).
Paraxial approximation for $x_m = 0$ (first part)

The Taylor series expansion in $\alpha$ of equation (43) takes the form

$$\left(\frac{v_0 t}{2}\right)^2 = R_{NIP}^2 + K^2 h^2 + \tan \alpha_0 R_{NIP} R_N (\alpha - \alpha_0) + \left\{1 - \frac{K R_{NIP}}{\cos \alpha_0} (1 + \cos^2 \alpha_0) -ight.$$  

$$K \frac{\sin \alpha_0}{\cos^2 \alpha_0} \left[ R_{NIP} + \cos^2 \alpha_0 (R_N - R_{NIP}) \right] (\alpha - \alpha_0) \right\} h^2 + O(2).$$

Observe that (44) may be recast as a second-order polynomial in $h^2$, whose admissible root is approximated as follows

$$h^2 = \frac{R_{NIP} R_N}{\sin \alpha_0 (R_{NIP} K - \cos \alpha_0)} (\alpha - \alpha_0) + O(2),$$

so that, after inversion,

$$\alpha - \alpha_0 = \frac{\sin \alpha_0 (R_{NIP} K - \cos \alpha_0)}{R_{NIP} R_N} h^2 + O(4).$$  

(46)
Paraxial approximation for $x_m = 0$ (second part)

Finally, after substitution of (46) into (45), we obtain the *hyperbolic* expression of the *time of flight*, as a function of the offset $h$, for a trajectory in a *homogeneous* medium near the reference NIP-ray:

$$
t^2(h) = \left( \frac{2R_{NIP}}{v_0} \right)^2 + \frac{4\cos \alpha_0}{v_0^2} (\cos \alpha_0 - 2KR_{NIP}) h^2 + O(4). \tag{47}
$$

From the power-series expansion of the square root of expression (47), we derive the *parabolic* form of the *time of flight*:

$$
t(h) = \frac{2R_{NIP}}{v_0} + \frac{\cos \alpha_0}{v_0 R_{NIP}} (\cos \alpha_0 - 2KR_{NIP}) h^2 + O(4). \tag{48}
$$

Equations (47) and (48) generalize equations (17) and (18) for an acquisition on a surface with curvature $\kappa = 2K$ at $x_0 = 0$, figure (13).
Curved surface, time of flight in homogeneous media: paraxial approximation

From the Taylor-series expansion (7), collecting (40) and (48), we obtain the parabolic approximation, valid near the reference NIP-ray in a constant-velocity media:

$$t(x_m, h) = \frac{2R_{NIP}}{v_0} + \frac{\cos \alpha_0}{v_0 R_{NIP}} (\cos \alpha_0 - 2KR_{NIP}) h^2 +$$

$$\frac{2 \sin \alpha_0}{v_0} x_m + \frac{\cos \alpha_0}{v_0 R_N} (\cos \alpha_0 - 4KR_N) x_m^2 + O(3). \quad (49)$$

Similarly, using (41) and (47), we derive the hyperbolic approximation:

$$t^2(x_m, h) = \frac{4}{v_0^2} R_{NIP}^2 + \frac{4 \cos \alpha_0}{v_0^2} (\cos \alpha_0 - 2KR_{NIP}) h^2 + 8 \frac{R_{NIP} \sin \alpha_0}{v_0^2} x_m +$$

$$\frac{4}{v_0^3} \left( \frac{R_{NIP} \cos^2 \alpha_0 + R_N \sin^2 \alpha_0}{R_N} - 2KR_{NIP} \cos \alpha_0 \right) x_m^2 + O(3). \quad (50)$$

Remark that setting $K = 0$, we recover the expressions obtained for a flat acquisition surface provided by equations (19) and (20).
Curved surface, time of flight in non-homogeneous media: paraxial approximation

The correction of the time of flight \( t(x_m, h) \) in a real medium requires the use of an auxiliary homogeneous model, leading again to the time delay (23). Therefore, from (49) and (50) we obtain the modified parabolic and hyperbolic expressions, valid for in-homogeneous media, which take into account the ground surface curvature:

\[
t(x_m, h) = t_0 + \frac{\cos \alpha_0}{v_0 R_{NIP}} (\cos \alpha_0 - 2KR_{NIP}) h^2 +
\]

\[
\frac{2 \sin \alpha_0}{v_0} x_m + \frac{\cos \alpha_0}{v_0 R_N} (\cos \alpha_0 - 4KR_N) x_m^2 + O(3), \tag{51}
\]

and

\[
[t(x_m, h) - (t_0 - \frac{2R_{NIP}}{v_0})]^2 =
\]

\[
\frac{4 R_{NIP}^2}{v_0^2} + \frac{4 \cos \alpha_0}{v_0^2} (\cos \alpha_0 - 2KR_{NIP}) h^2 +
\]

\[
\frac{8R_{NIP} \sin \alpha_0}{v_0^2} x_m + \frac{4}{v_0^2} \left( \frac{R_{NIP} \cos^2 \alpha_0 + R_N \sin^2 \alpha_0}{R_N} - 2KR_{NIP} \cos \alpha_0 \right) x_m^2 + O(3). \tag{52}
\]
General conclusion

The seismic reflection-time derivation presented in this document provides an analytic solution independent of the subsurface macro-velocity model. The resulting expressions are defined, for an arbitrary source-receiver pair, in the midpoint-offset domain.

To reach this goal, firstly, we started with the homogeneous problem where this derivation was based on a paraxial approximation of $t(x_m, h)$, the reflection time, assuming down-going and up-going ray trajectories near a reference NIP-ray. This last trajectory was characterized by three parameters denoted as $\alpha_0$, $R_{NIP}$ and $R_N$.

Secondly, a homogeneous subsurface reference model was designed to behave as the real medium under examination, in the sense that NIP- and N-waves, and their images in the reference world reach the surface with different time of flight but with the same radii of curvature, $R_{NIP}$ and $R_N$, and emergence angle $\alpha_0$. In addition, these waves are assumed to propagate with the same velocity near the surface.

Consequently, close to the surface, traveled distances are the same in both models, a feature which provides the estimate of the time correction $\Delta t(x_m, h)$ between the true non-homogeneous medium and the reference one, leading to the proper expression of time of flight $t(x_m, h)$ as a function of the three parameters $\alpha_0$, $R_{NIP}$ and $R_N$. 
References


