Arbitrary Lagrangian Eulerian (ALE) Formulation. Application for Euler and Navier-Stokes equations

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Abstract

When fluid simulation has to deal with moving boundary domains there is a problem when one wants to make finite differences in time at points where the fluid is not always present. One solution for this is to transport the equation on a fixed domain. We present here the ALE derived formulation consisting to fix the domain only for one time step. We give the derivations allowing to reformulate the Euler equation with obvious extensions to a wide range of PDE.

1 Introduction

Our aim is to solve the incompressible NSE for moving domain. We are going to use concepts issued from the Arbitrary Lagrangian Eulerian formulation. An extensive presentation of its basic concept can be found in [5]. For a clear and concise description of the ALE formulation, see [1]. For simplicity, we restrain to the Euler corresponding equation.
Find $(u(x,t), P(x,t)) \in \Omega(t) \times [-\Delta t, +\Delta t]$, solution of

$$\left\{ \begin{array}{ll} \partial_t u + (u.\nabla)u + \nabla P = 0 & \text{in } \Omega(t) \\ \nabla.u = 0 & \end{array} \right. \tag{1}$$

The Ale formulation is:

$$\partial_t \int_{\Omega_i(t)} u + \int_{\Gamma_i(t)} (uu - cu + PI).n = 0 \qquad \forall \Omega_i(t) \subset \Omega(t). \tag{2}$$

In the former expression, I is the unit matrix and c is the velocity of the boundary $\Gamma_i(t) = \partial\Omega_i(t)$.
We want to transport the equations (1) on $\Omega(0)$. We need a continuous family of mapping from each $\Omega(t)$ towards $\Omega(0)$. This is done doing the change of variables:

$$(x,t) \longmapsto (y,\tau)$$

where

$$\left\{ \begin{array}{l} \tau = t \\ y(x,t) = x - M(x,t) \end{array} \right. \tag{3}$$

such that $y(\Omega(t), t) = \Omega(0)$.
We are now able to define three useful quantities:

$$\left\{ \begin{array}{l} N(y,\tau) = M(x,t) \\ v(y,\tau) = u(x,t) \\ Q(y,\tau) = P(x,t). \end{array} \right. \tag{4}$$

$N(y,\tau)$ is the displacement of the domain vue from the $\Omega(0)$ point of view. The "velocity" of the domain would then be given by $\partial_\tau N$.
We are going to write the system (1) in the (v,τ) variables.

2 Preliminary calculations

The composition differentiation rule gives:

$$\left\{ \begin{array}{l} \partial_t u = \partial_\tau v + (\partial_t y.\nabla_y)v \\ \nabla_x u = \nabla_x y.\nabla_y v \\ \nabla_x P = \nabla_x y.\nabla_y Q. \end{array} \right. \tag{5}$$

So we have to evaluate $\partial_t y$ and $\nabla_x y$ in the new variables:

$$\left\{\begin{array}{l} \nabla_x y = I - \nabla_x M = I - \nabla_x N = I + \nabla_x y.\nabla_y N \\ \nabla_x y(I + \nabla_y N) = I \\ \nabla_x y = (I + \nabla_y N)^{-1} \end{array}\right. \tag{6}$$

where I is the identity matrix.

$$\left\{\begin{array}{l} \partial_t y = -\partial_t M = -\partial_t N = -\partial_\tau N - \partial_t y.\nabla_y N \\ \partial_t y(I + \nabla_y N) = -\partial_\tau N \\ \partial_t y = -\partial_\tau N(I + \nabla_y N)^{-1}. \end{array}\right. \tag{7}$$

3 New formula

From the preceding derivations, we get:

$$\left\{\begin{array}{l} \partial_t u = \partial_\tau v - (\partial_\tau N(I + \nabla_y N)^{-1}.\nabla_y)v \\ \nabla_x u = (I + \nabla_y N)^{-1}\nabla_y v \\ \nabla_x P = (I + \nabla_y N)^{-1}\nabla_y Q. \end{array}\right. \tag{8}$$

So we have:

$$\left\{\begin{array}{l} \partial_\tau v - (\partial_\tau N(I + \nabla_y N)^{-1}.\nabla_y)v + (v(I + \nabla_y N)^{-1}\nabla_y)v + (I + \nabla_y N)^{-1}\nabla_y Q = 0 \\ Trace[(I + \nabla_y N)^{-1}\nabla_y v] = 0. \end{array}\right. \tag{9}$$

4 Special case

In the special case for which $N(y, \tau) = \tau.c(y)$, that is when the "backward" mapping is done at constant velocity, we get the following system:

$$\left\{\begin{array}{l} \partial_\tau v - (c.(I + \tau\nabla c)^{-1}.\nabla)v + (v(I + \tau\nabla c)^{-1}\nabla)v + (I + \tau\nabla c)^{-1}\nabla Q = 0 \\ Trace[(I + \tau\nabla c)^{-1}\nabla v] = 0. \end{array}\right. \tag{10}$$

Neglecting the terms in $\mathcal{O}(\tau)$, we get the first order approximating system:

$$\left\{\begin{array}{l} \partial_\tau v + [(v - c)\nabla)]v + \nabla Q = 0 \\ \nabla.v = 0. \end{array}\right. \tag{11}$$

When $|\tau\nabla c|$ is much smaller than 1, we can develop $(I + \tau\nabla c)^{-1}$ in a series expansion:

$$(I + \tau\nabla c)^{-1} = \sum_{i=0}^{\infty}(-\tau)^i\nabla c^i, \tag{12}$$

to obtain any higher order approximation. The second order approximation allows us to evaluate the error of the first order one. It comes:

$$\left\{\begin{array}{l} \partial_\tau v - (c.(I - \tau\nabla c).\nabla)v + (v(I - \tau\nabla c)\nabla)v + (I - \tau\nabla c)\nabla Q = 0 \\ \nabla.v = \tau Trace[\nabla c.\nabla v]. \end{array}\right. \tag{13}$$

Differentiating in time this last divergence equation we get at time $\tau = 0$:

$$\nabla.v'_{|\tau=0} = Trace[\nabla c.\nabla v]_{|\tau=0} \tag{14}$$

which is potentially far from being zero. That is the reason why one should be very cautious when using an algorithm where it is implied that $\partial_t \nabla.v = 0$ as should be inferred from (11).

5 Navier-Stokes extension

To obtain the incompressible Navier-Stokes equations, we have to evaluate the diffusive term corresponding to $-\nu\Delta u$. The differentiation rule gives a rather complicated formula, which rend the "highly" contracted notations necessary:

$$\Delta_x u = -A_{il}A_{im}N_{n,lm}A_{nk}v_{,k} + A_{il}A_{ik}v_{,lk} \tag{15}$$

with $A = (I + \nabla N)^{-1}$.
Again, with a series expansion of A, we get at first order:

$$\Delta u = \Delta v - (\Delta N.\nabla)v + \mathcal{O}(\nabla N) \tag{16}$$

and at second order:

$$\Delta u = \Delta v - (\Delta N.\nabla)v + N_{k,l}(-2v_{,kl} + 2N_{j,kl}v_{,j} + N_{l,jj}v_{,k}) + \mathcal{O}(\nabla N)^2 \tag{17}$$

For the special case in which we can separate the variables of N, i.e. when $N(y,\tau) = \tau.c(y)$, we can compute c by solving a Poisson problem, as it is done in N3S:

$$\begin{cases} \Delta c = 0 & \text{in } \Omega(0) \\ c(\Gamma(0)) = c_0. & \end{cases} \tag{18}$$

Then we get more simple approximations:

$$\begin{cases} \Delta u &= \Delta v + \mathcal{O}(\nabla N) \\ &= \Delta v + \mathcal{O}(\tau) \\ \Delta u &= \Delta v + N_{k,l}(-2v_{,kl} + 2N_{j,kl}v_{,j}) + \mathcal{O}(\nabla N)^2 \\ &= \Delta v - 2.\tau\nabla c.\nabla\nabla v + \mathcal{O}(\tau^2) \\ &= (I - \tau\nabla c)^2.\nabla\nabla v + \mathcal{O}(\tau^2). \end{cases} \tag{19}$$

In fact this is still true for N of the form $g(\tau).c(y)$ so that acceleration could be taken into account and in this case $\partial_\tau g$ must be present in the transport term. Things become more complicated when one wants to propagate the boundary not only along straight segments but also along curves, this is the case for which separation of variables is impossible for N.

6 Conclusion

In [4], a first order approximation of the ALE formulation has been derived using the domain velocity as the main parameter. Using a mapping family, we have derived an ALE formulation that is exact before discretization. High order approximations can be easily written and the consistency error is clearly seen and controllable by a CFL-type parameter. The first order approximation has been quite successfully implemented in N3S, using either the $[-\Delta t, 0]$ [3] or the $[0, \Delta t]$ [2] time interval. Beautiful results are obtained when the movement of the boundary is prescribed. It seems to be a good candidate for a coupling with a structure across the boundary conditions.

7 Annex: semi-integral Ale formulation for finite elements and finite volumes

The strategy used in the first part has been to transport the equation on a fixed domain. We are giving here another one which use the specific properties of the control volumes (for finite volumes) and the test functions (for finite elements) for an evolving mesh. Using the previous notations, we are interested at what happends at time $t = 0$. Let $\phi(x, t)$ be either the characteristic function of a control volume or a test function, evolving with time. Its principal property is to be time invariant in the frame of reference moving with the mesh (or the mapping in the continuous case). That is:

$$\frac{d\phi}{d\tau} = 0$$

which gives at time $t = 0$:

$$c.\nabla\phi + \partial_t\phi = 0 \tag{20}$$

with $c(x, 0) = -\lim_{t\to 0} \partial_t M(x, t)$.
We want to evaluate the quantity:

$$X = \int_{\Omega(t=0)} \partial_t u.\phi \tag{21}$$

We call ω the support of ϕ and γ its boundary. We have:

$$\begin{aligned}
X &= \int_{\omega(t=0)} \partial_t u.\phi && (22)\\
&= \int_{\omega(t=0)} \partial_t(u.\phi) - u.\partial_t\phi && (23)\\
&= \int_{\omega(t=0)} \partial_t(u.\phi) + u.(c.\nabla)\phi && (24)\\
&= [\partial_t \int_{\omega(t)} (u.\phi)]_{t=0} - \int_{\gamma(t=0)} u\phi.c.n + \int_{\omega(t=0)} u.(c.\nabla)\phi && (25)\\
&= [\partial_t \int_{\omega(t)} (u.\phi)]_{t=0} - \int_{\omega(t=0)} \nabla_i(c_i.u)\phi && (26)\\
&= [\partial_t \int_{\Omega(t)} (u.\phi)]_{t=0} - \int_{\Omega(t=0)} \nabla_i(c_i.u)\phi. && (27)
\end{aligned}$$

Time $t=0$ was only a convenience and the formula is valid for all t at the cost of naturally propagating the definition of c for $t \neq 0$. Finally, we obtain:

$$\partial_t \int_{\Omega(t)} (u.\phi) + \int_{\Omega(t)} (\nabla_i[(u_i - c_i).u] - \nu\Delta u + \nabla P).\phi = 0 \qquad (28)$$

This formula is valid for all ϕ which shape is following the domain movement. When ϕ is the characteristic function of a control domain, we find back the original ALE-formulation (2) so equation (28) is e generalization of it (a GALE-formulation).It has the advantage of solving the conflicts that arise when one wants to advance in time without multiplying test functions that live at different times. The idea presented here is somewhat inspired from the works presented by C.Farhat and M.Lesoinne at the Cemracs School of July 1996 (see [6],[7]). (These works where concerned with the implications of the Geometrical Conservation Law (GCL) for moving domains.)

An algorithm based on this formulation could be an alternative at the one used in the N3S software.

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