Pressure-related algorithm for fluid structure interaction.

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1 Introduction

Coupling fluid and structure equations is not a simple task, specially when the fluid is considered incompressible. A first problem is that the main variable is not the same at the interface for the fluid and the structure. The main fluid variable is its velocity while the main structure variable is its position. Moreover the coupling is essentially done by equalizing the dual stress variable, which is essentially the pressure which acts as a Lagrange multiplier for the incompressible fluid equation. Both side of the interface are evolving with different and incompatible laws so it is impossible to satisfy at the same time and all the time the equality of all the relevant quantities: position, velocity, acceleration and normal stress. This work is an improvement and in direct continuation of Jaubert & Moreau (1997) ([3]) with essentially the same notations. In this paper, we will try to use the fluid pressure indetermination and the fluid evolution algorithm to match the evolution of the interface.

2 Continuous equations

The continuous fluid equation is given by:

$$\begin{cases} \partial_t u + (u_i - c_i)\nabla_i u - \nu\Delta u + \nabla P) = 0 \\ div(u) = 0 \end{cases} \tag{1}$$

Where u is the ALE fluid velocity, i. e. the fluid velocity at a location moving with velocity c, and P is the pressure, normalised with the fluid density. The equation is valid on an evolving domain $\Omega(t)$. The structure equation is of the form:

$$\eta'' + \alpha\eta' + \kappa\eta = \gamma P \tag{2}$$

where η is the displacement of the structure, η' its velocity and η'' its acceleration. The displacement is supposed small and normal to the boundary. Changes of the normal with the displacement are neglected.

The interface condition reads:

$$u.n = \eta' \tag{3}$$

with n the unitary vector normal to the fluid boundary and externally oriented. Another hidden condition is that the boundary of the fluid lies on on the one-dimensional structure. Normally the right-hand-side of (2) should have a viscous stress component. We neglect it here to simplify the discussion.

3 Effects of the fluid algorithm

The advancement in time of the fluid part is done by a splitting method coupled with the ALE formulation. In fact, one time step is thought to be solved over a moving domain, but the velocity of the boundary is taken constant in time over the time step. So, if the boundary velocity changes, the change is approximated by a function which is discontinuous in time. If the boundary geometry is not plane, curvature effects forbid to respect the compatibility condition: $\int_\Omega u.n = 0$ both at the beginning and at the end of the time step as the displacement of the boundary is equal to the integral of its velocity, constant over the time step. From the structure point of view, the interface conditions force the structure acceleration to be a bounded measure, with a Dirac time component at each change of time step. If we want the position to be continuous in time, we are forced to consider that the applied pressure has the same structure of the boundary acceleration, which includes the Dirac time component at each change of time step. In other words, the only way to change the velocity of the boundary without changing its position is to apply a Dirac of pressure. Moreover, we can fix this Dirac of pressure so as to satisfy the hidden compatibility condition at the beginning of one time step (on the old geometry). The application of the "L^2"part of the pressure would allow to satisfy the compatibility condition at the end of the time step on the new geometry. We are now going to describe a way to do that.

4 Interface algorithm

In this part we suppose that the boundary is composed of a flow inlet Γ_1 and a moving boundary Γ_2. To fix the ideas, we can consider an elastic pipe. Suppose that we know all relevant quantities at time n. We first have to compute the new position of the moving boundary. For this purpose, we suppose the numerical pressure P to be split-able in the 3 following parts:

$$P(x,t) = S(i).\delta(i.Dt) + C(t) + Q(x,t) \tag{4}$$

where $i \in \mathbf{N}$.The variable C is piecewise constant in time and Q is the pressure deriving from the fluid computation. We have used δ for the Dirac measure.
From (4) and integrating (2), it follows that:

$$[\eta']_{n-}^{n+} = \gamma.S(n) \tag{5}$$

with the preceding left-hand-side standing for the jump of velocity at time $n.Dt$. The application of the compatibility condition $\int_\Gamma u.n = 0$ at "time" t^{n+} produces:

$$S(n).\int_{\Gamma_{2(n-)}} \gamma + \int_{\Gamma_1} [u.n]_{n-}^{n+} = 0 \tag{6}$$

which defines $S(n)$ one way:

$$S(n) = -\frac{\int_{\Gamma_1} [u.n]_{n-}^{n+}}{\int_{\Gamma_{2(n-)}} \gamma}. \tag{7}$$

We now have to determine the value of C. While it should be related in some way to $S(n)$, we must stress the fact that it reflects also the evolution of the fluid over the entire time step and so has also a mean value signification. That is why we disconnect the two computations. The idea under the determination of C is the following: the moving wall evolving over the time step $(n.Dt, (n+1).Dt)$ following the equation (2) should move in such a way that the compatibility condition should be satisfied at "time" $t^{(n+1)^-}$.
It turns out that the equation verified by C is highly related to the way we discretized the structure equation, as we show in the next section.

5 Structure equation and its resolution

We choose to discretize (2) with finite differences (only for simplicity). We do the time discretizetion in the following way:

1. $\eta'' = \frac{\eta'^{n+1} - \eta'^n}{Dt}$
2. $\eta = \beta_1 \eta^{n+1} + (1-\beta_1)\eta^n$
3. $\eta' = \beta_3 \eta'^{n+1} + (1-\beta_3)\eta'^n$
4. $\eta^{n+1} = \eta^n + Dt(\beta_2 \eta'^{n+1} + (1-\beta_2)\eta'^n)$

The data are: η^n and η'^n. The unknown are: η^{n+1} and η'^{n+1}. The constants β_i, $i \in \{1,2,3\}$ have value in $[0,1]$.
With this discretization, we obtain the following explicit formula:

$$\eta'^{n+1} = \frac{Dt.(\gamma.P - \kappa.\eta^n) + \eta'^n.[1 - \alpha(1-\beta_3).Dt - \kappa.\beta_1.(1-\beta_2).Dt^2]}{1 + Dt.\alpha.\beta_3 + Dt^2.\kappa.\beta_1.\beta_2} \tag{8}$$

$$\eta^{n+1} = \eta^n + \frac{Dt.\eta'^n + Dt^2.\beta_2.(\gamma.P - \kappa.\eta^n) + Dt^2.\alpha.(\beta_3 - \beta_2).\eta' n}{1 + Dt.\alpha.\beta_3 + Dt^2.\kappa.\beta_1.\beta_2} \tag{9}$$

We will note by $\tilde{\eta'}$ the right-hand-side of (8) with the pressure P changed by its component coming from the fluid discretization Q. We write "$\mathcal{D}$" for the quantity under the bar-fraction:

$$\mathcal{D} = 1 + Dt.\alpha.\beta_3 + Dt^2.\kappa.\beta_1.\beta_2.$$

Then, applying the compatibility condition at time $t^{(n+1)^-}$ and ricording that $P^{(n+1)^-} = Q + C^{(n+\frac{1}{2})}$, it comes:

$$\int_{\Gamma_1} u^{n+1-}.n + \int_{\Gamma_2(n+1)} \tilde{\eta'} + C^{(n+\frac{1}{2})}.\int_{\Gamma_2(n+1)} \frac{Dt.\gamma}{\mathcal{D}} = 0 \tag{10}$$

or

$$C^{(n+\frac{1}{2})} = -[\int_{\Gamma_2(n+1)} \frac{Dt.\gamma}{\mathcal{D}}]^{-1}.(\int_{\Gamma_1} u^{n+1-}.n + \int_{\Gamma_2(n+1)} \tilde{\eta}) \tag{11}$$

Unfortunately, this formulae is not sufficient to calculate C because the right-hand-side of (11) depends itself of C. Nevertheless, it can be inferred from (8) and (9) that the position and velocity of the boundary are affine functions of C with relative slopes showing a ratio of Dt. This mean that for small time steps, the velocity of the boundary is much more sensitive to C than its position. So it can be expected that in normal circumstances the integrals in (11) are not very sensitive to C. This quantity should then be easy to find using a fixed point based on (11) with $\tilde{\eta'}$ converging to η'.
While this way of proceeding could seem reasonable, it should be stressed the fact that the normal velocity of the boundary is not equal to the domain velocity.
In the following we rewrite the equations (8) and (9) for all β_i parameters equal to one, which is fully time implicit:

$$\begin{cases} \eta'^{n+1} = \dfrac{Dt.(\gamma.P - \kappa.\eta^n) + \eta'^n}{1 + Dt.\alpha + Dt^2.\kappa} \\ \eta^{n+1} = \eta^n + \dfrac{Dt.\eta'^n + Dt^2.(\gamma.P - \kappa.\eta^n)}{1 + Dt.\alpha + Dt^2.\kappa} \end{cases} \tag{12}$$

In fact the second formulae of (12), can be rewritten in a more simple fashion:

$$\eta^{n+1} = \eta^n + Dt.\eta'^{n+1} \tag{13}$$

6 Priminary results

This algorithm has been implemented for the simulation of the aortic arch. It was made necessary because the simulation shown an explosive instability at the change of curvature of the inlet fluid velocity time evolution. More precisely the instability developed when the fluid acceleration in inlet started to decrease. This kind of instability was first reported by I.Heude ([1])in a quite similar context. The result was optimum and no more instability appears. Nevertheless, the external model, ie the modeling of the external part of the fluid computational domain, was no more satisfying. It was modeled as a solid tube with friction boundary effects. Because the (assimilated) solid boundary parts react with their relative mass to the jump of velocity, this modeling strongly underestimate the outflow. An alterate external boundary model which takes into account the three-parts-structure of the pressure is now under study.

7 conclusion

Fluid-structure interactions show strong instabilities when the fluid is considered incompressible and solved by a projection plus splitting method. This arises as soon has the time steps goes smaller to the structure elastic characteristic time, which is around one millisecond for the aortic arch. While a first instability due to boundary acceleration was stabilized by a separation of time step procedure (see [2]), we have stabilized another instability due to the decreasing of the inlet boundary acceleration. The stabilization relies upon physical considerations about the fluid algorithm. It should be interesting to discover the mathematical causes of these instabilities, and to justify theoretically the consistency of the algorithm.

8 Appendice: the external model

As already said, we are using an external simplified model for our simulation of the aortic arch. The key point is to determine for a given input flow, what proportion in going in outflow ant what proportion is used to modify the inside volume via the boundary position evolution. Six variables are thought to be taken into consideration:

1. viscous dissipation,
2. compliance of the arteries,
3. fluid inertia,
4. fluid velocity,
5. pressure drop,
6. topology of the arterial system.

Simplified models can be done using an electrical analogy. To the six points just given, we associate the following quantities:

1. resistance in series,
2. capacity in derivation,
3. self-inductance in series,
4. intensity,
5. electrical potential,
6. electrical net.

By adding elements, arbitrarily complicated and accurate external models can be simulated.

We present now a simple external model more fluid oriented.

We think the outlet part of the aorta as a compliant tube W_{ext} of length L with the same properties of the aorta near the Neumann boundary condition. We suppose that the wall velocity evolve linearly along the tube and that the pressure drop is such that the distal pressure is constant (at zero). We also make the assumption that the outlet flow is constant, this constant being the mean temporal value of the inlet flow. By continuity of the pressure, the integral of the normal wall velocity is given by its mean values at the two-points interface $M_{ini,ext}, M_{ini,int}$, with the fully simulated arterial wall.

$$\int_{W_{ext}} \eta' = L\dot{\{}\eta'(ini) + \eta'(fin)\} \tag{14}$$

In this modeling, the space-constant correction is in fact decreasing linearly to zero in the external domain, because the pressure at the final outlet is always considered to be zero, even after correction. We can then adapt the computing of (11). Splitting the boundary Γ_2 in two parts: the one effectively discretized called Γ_{dis} and the external part Γ_{ext}, it comes:

$$C(n+\frac{1}{2}) = -[L\frac{D\dot{t}.\gamma}{\mathcal{D}} + \int_{\Gamma_{dis}(n+1)} \frac{Dt.\gamma}{\mathcal{D}}]^{-1}.(L\dot{\{}\eta'(ini) + \eta'(fin)\} + \int_{\Gamma_1} u^{n+1-}.n + \int_{\Gamma_{dis}(n+1)} \tilde{\eta}) \tag{15}$$

9 bibliography

References

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[3] D. Jaubert & V. Moreau *Fluid-structure interaction: Applications in the context of a flow in arterial medium CRS4 internal report: CRS4-TECH-REP97/15 February 97, and also Cemracs96 report*