Coercive Domain Decomposition Algorithms for Advection-Diffusion Equations and Systems

Ana ALONSO$^{(*)}$**, R. Loredana TROTTA**$^{(**)}$
and Alberto VALLI$^{(*)}$

Summary. Two families of non-overlapping coercive domain decomposition methods are proposed for the numerical approximation of advection dominated advection-diffusion equations and systems. Convergence is proven for both the continuous and the discrete problem. The rate of convergence of the first method is shown to be independent of the number of degrees of freedom. Several numerical results are presented, showing the efficiency and robustness of the proposed iterative algorithms.

1. Introduction

The interest for the use of domain decomposition methods for advection-diffusion equations has considerably grown in the last years (see, e.g., [14], [16], [7], [11], [17], [8], [9], [2] for non-overlapping partitions, [6], [18] for overlapping partitions).

In this paper we are concerned with *non-overlapping* domain decomposition methods for advection dominated advection-diffusion equations and systems. The computational domain Ω, a connected open bounded subset of $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$, will be split into two non-overlapping subdomains Ω_1 and Ω_2. We set $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$, and denote by $\mathbf{n}$ the unit normal vector on Γ, directed from Ω_1 to Ω_2.

We propose two families of methods, depending on the choice of a parameter, denoted by γ, and show their convergence, for both the continuous problem and its discrete approximation. The first method, called γ-DR, turns out to have a rate of convergence which is independent of the mesh size h, hence it introduces an *optimal* preconditioner for the associated Schur complement matrix related to the unknown nodal values on the interface Γ.

The main novelty in our methods resides in the fact that we don't care about the local direction of the advective field $\mathbf{b}$ on Γ (as in adaptive methods proposed in [7], [11]), but we only need that the boundary value problems in Ω_1 and Ω_2 along the subdomain iterations are associated to a suitable *coercive* bilinear form.

To start with, in Sections 2, 3 and 4 we consider the following homogeneous

$^{(*)}$ Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy. Telefax: 39-461-881624. E-mail: alonso@science.unitn.it, valli@science.unitn.it
$^{(**)}$ CRS4-CFD, via Nazario Sauro 10, 09123 Cagliari, Italy. Telefax: 39-70-2796245. E-mail: loredana@crs4.it

Dirichlet boundary value problem

$$
(1.1) \qquad \begin{cases} L_\varepsilon u := -\varepsilon \Delta u + \displaystyle\sum_{j=1}^{d} D_j(b_j u) + a_0 u = f \quad \text{in } \Omega \\ \\ u_{|\partial\Omega} = 0 \ , \end{cases}
$$

where D_j denotes the derivative with respect to x_j, $j = 1, ..., d$, $f \in L^2(\Omega)$ and the coefficients satisfy the regularity conditions

$$
\mathbf{b} \in (L^\infty(\Omega))^d \ , \ \operatorname{div} \mathbf{b} \in L^\infty(\Omega) \ , \ a_0 \in L^\infty(\Omega)
$$

and the coerciveness condition

$$
(1.2) \qquad \frac{1}{2} \operatorname{div} \mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq 0 \qquad \text{for almost each } \mathbf{x} \in \Omega \ .
$$

As a consequence of (1.2), the associated bilinear form

$$
(1.3) \qquad \begin{aligned} a^\sharp(w, v) := & \int_\Omega \left[\varepsilon \nabla w \cdot \nabla v + \left(\frac{1}{2} \operatorname{div} \mathbf{b} + a_0 \right) w\, v \right] \\ & + \frac{1}{2} \int_\Omega (v\, \mathbf{b} \cdot \nabla w - w\, \mathbf{b} \cdot \nabla v) \ , \end{aligned}
$$

is continuous and coercive in $V = H_0^1(\Omega)$, the Sobolev space of functions belonging to $L^2(\Omega)$ together with their first order distributional derivatives.

The variational formulation of (1.1) reads:

$$
(1.4) \qquad \text{find } u \in H_0^1(\Omega) \ : \ a^\sharp(u, v) = \int_\Omega f v \qquad \forall \, v \in H_0^1(\Omega) \ .
$$

The Lax-Milgram lemma ensures that the solution to (1.4) exists and is unique.

The results we are going to present can be straightforwardly extended to other boundary conditions, provided that the coerciveness of the associated bilinear form is still satisfied.

In Section 5 we will take into consideration the case of systems of advection dominated advection-diffusion equations. The extension of the proposed methods to this case turns out to be an easy task. On the contrary, it is worthwhile to notice that this is not the case for the adaptive methods devised in [7], [11], as these algorithms are based on the knowledge of the direction of the flow on the interface Γ, and this information is not easily available for systems of advection-diffusion equations.

Finally, the numerical results illustrating the performances of the proposed methods are presented in Section 6, for several suitable benchmark problems. The γ-DR method turns out to be very efficient and robust, and the numerical examples show that the choice of the parameter γ and of the relaxation coefficient θ (see (3.2) and (3.3), respectively) can be done in a simple way. In conclusion, these results suggest to propose the γ-DR method as an "universal" non-overlapping domain decomposition procedure for advection-diffusion equations and systems.

2. A model one-dimensional problem

Let us start by considering the model problem

$$
\begin{cases} L_\varepsilon u = -\varepsilon u_{xx} + b u_x + a_0 u = f \quad \text{in } \Omega = (0,1) \\ \\ u(0) = u(1) = 0 \ , \end{cases} \tag{2.1}
$$

where $\varepsilon > 0$, $b \neq 0$ and $a_0 \geq 0$ are constant coefficients.

In [11] the following iteration-by-subdomain method to solve (2.1) has been analyzed: given λ^0, solve for $k \geq 1$

$$
\begin{cases} L_\varepsilon u_1^{k+1} = f \qquad \text{in } \Omega_1 = (0,c) \\ \\ u_1^{k+1}(0) = 0 \\ \\ \varepsilon u_{1x}^{k+1}(c) - (\frac{1}{2}b + A) u_1^{k+1}(c) = \lambda^k \ , \end{cases} \tag{2.2}
$$

then

$$
\begin{cases} L_\varepsilon u_2^{k+1} = f \qquad \text{in } \Omega_2 = (c,1) \\ \\ u_2^{k+1}(1) = 0 \\ \\ \varepsilon u_{2x}^{k+1}(c) - (\frac{1}{2}b + B) u_2^{k+1}(c) = \varepsilon u_{1x}^{k+1}(c) - (\frac{1}{2}b + B) u_1^{k+1}(c) \ , \end{cases} \tag{2.3}
$$

and finally set

$$
\lambda^{k+1} = \varepsilon u_{2x}^{k+1}(c) - \left(\frac{1}{2}b + A\right) u_2^{k+1}(c) \ , \tag{2.4}
$$

where $0 < c < 1$, and A and B are real parameters, with $A \neq B$. Indeed, the cases $A = \pm\infty$ and $B \in \mathbf{R}$, or $A \in \mathbf{R}$ and $B = \pm\infty$ can also be considered. For these choices one of the two first-order interface conditions in c becomes a Dirichlet boundary condition.

The convergence of this method is achieved provided that

$$
|\rho_\varepsilon(A,B)| < 1 \ , \tag{2.5}
$$

where

$$
\rho_\varepsilon(A,B) := \frac{\tau \coth(\tau c) - B/\varepsilon}{\tau \coth(\tau c) - A/\varepsilon} \, \frac{\tau \coth[\tau(1-c)] + A/\varepsilon}{\tau \coth[\tau(1-c)] + B/\varepsilon} \ ,
$$

and

$$
\tau := \frac{\sqrt{b^2 + 4\varepsilon a_0}}{2\varepsilon}
$$

(see [11], Section 3, in which $\alpha = \frac{1}{2}\,b + A$ and $\beta = \frac{1}{2}\,b + B$).

Introducing a relaxation parameter $\theta \neq 0$, we can consider a more general iterative scheme in which

$$\lambda^{k+1} = \theta \left[\varepsilon u_{2,x}^{k+1}(c) - \left(\frac{1}{2}\,b + A \right) u_2^{k+1}(c) \right] + (1-\theta)\lambda^k \ . \tag{2.6}$$

In this case we have convergence when

$$|1 - \theta[1 - \rho_\varepsilon(A,B)]| < 1 \ . \tag{2.7}$$

This means

$$\begin{cases} 0 < \theta < \frac{2}{1-\rho_\varepsilon(A,B)} & \text{for } \rho_\varepsilon(A,B) < 1 \\ \frac{2}{1-\rho_\varepsilon(A,B)} < \theta < 0 & \text{for } \rho_\varepsilon(A,B) > 1 \end{cases} . \tag{2.8}$$

(Notice that $\rho_\varepsilon(A,B) \neq 1$ for $A \neq B$.)

The iterative method based on the relaxation procedure (2.6) is therefore convergent, provided we choose θ as in (2.8). However, we are interested in advection-dominated problems, namely, the "viscous" parameter ε we are considering is very small in comparison with b and a_0. An efficient method in this situation is therefore the one which converges for a choice of θ independent of ε as $\varepsilon \to 0^+$.

A direct calculation shows that

$$\rho_0(A,B) := \lim_{\varepsilon \to 0^+} \rho_\varepsilon(A,B) = \frac{|b|/2 - B}{|b|/2 + B}\,\frac{|b|/2 + A}{|b|/2 - A} \ , \tag{2.9}$$

hence the choices $A = |b|/2$ and $B = -|b|/2$ lead to a non-efficient scheme. These correspond to imposing the value of the normal derivative on the inflow region, or the value of the conormal derivative on the outflow region. Notice that when we consider these boundary conditions the boundary value problem at hand is associated to a non-coercive bilinear form.

When the asymptotic reduction factor $\rho_0(A,B)$ belongs to the interval $(-1,1)$, the relaxation parameter can be chosen in the whole interval $(0,1]$, leading to efficient iterative schemes. By means of a simple computation one can see that the values of the parameters A and B for which $-1 < \rho_0(A,B) < 1$ strictly contains the region

$$\mathcal{C} := \{(A,B) \in \mathbf{R}^2 \,|\, A \leq 0 \ , \ B \geq 0 \ , \ A \neq B\} \ . \tag{2.10}$$

More precisely, choosing (A^*, B^*) in the region of convergence (i.e., where $-1 < \rho_0(A,B) < 1$) but not in $\mathcal{C}$, the absolute value of exactly one of the two factors in (2.9) is strictly larger than one. In this situation, we have either $A^* > 0$ or $B^* < 0$. To fix the ideas, suppose that $B^* < 0$. The argument above says that we can improve the rate of convergence of the iterative scheme by only changing the interface condition in Ω_2, substituting the one associated to B^* with another one, related to any parameter $B \geq 0$, i.e., choosing (A^*, B) in the region $\mathcal{C}$. Therefore, one should expect best convergence properties choosing the parameters in $\mathcal{C}$.

The region $\mathcal{C}$ is exactly the set of parameters A and B for which both the bilinear forms, associated to the boundary value problems we are considering, are coercive for each choice of the ellipticity coefficient ε. The limit cases $A = -\infty$, $B \geq 0$ (Dirichlet boundary condition in Ω_1) and $B = \infty$, $A \leq 0$ (Dirichlet boundary condition in Ω_2) can be also included.

The analysis performed in [11] leaded the authors to propose adaptive iterative schemes for advection-dominated advection-diffusion equations. In this context, adaptivity means that the boundary conditions imposed along the iterations are consistent with the "hyperbolic" limit as $\varepsilon \to 0^+$, namely, the Dirichlet boundary condition is never imposed on the outflow. In fact, this choice could create artificial internal layers at the interface.

We are going to suggest here a different point of view. The choice that leads to efficient iteration-by-subdomain schemes is the one which, in each subdomain, preserves the *coerciveness* of the associated bilinear forms. For example, the Dirichlet boundary condition can always be imposed, no matter if the interface is an inflow or an outflow boundary. Numerical evidence will show that the artificial internal layers, which indeed arise, are damped out after very few iterations, provided that the relaxation parameter θ is suitably chosen, and don't affect convergence in a significative way. To illustrate this behaviour, we present Fig. 2.1 and Fig. 2.2. In both figures, we refer on the left to the equation (2.1) for $\varepsilon = 10^{-2}$, $b = 1$, $a_0 = f = 0$, and on the right to the same case, but with boundary condition $u(0) = 1$. In Fig. 2.1 the splitting (2.2)-(2.4) have been performed with $A = -\infty$, $B = \frac{1}{2}b$, i.e., we consider to the so-called Dirichlet/Robin scheme, and in Fig. 2.2 we have taken $A = -\infty$, $B = 0$, which corresponds to the 0-DR scheme we are going to propose in Section 3. The optimal choice of θ is clearly $\theta = 1$ in Fig. 2.1 and $\theta = 0.5$ in Fig. 2.2.

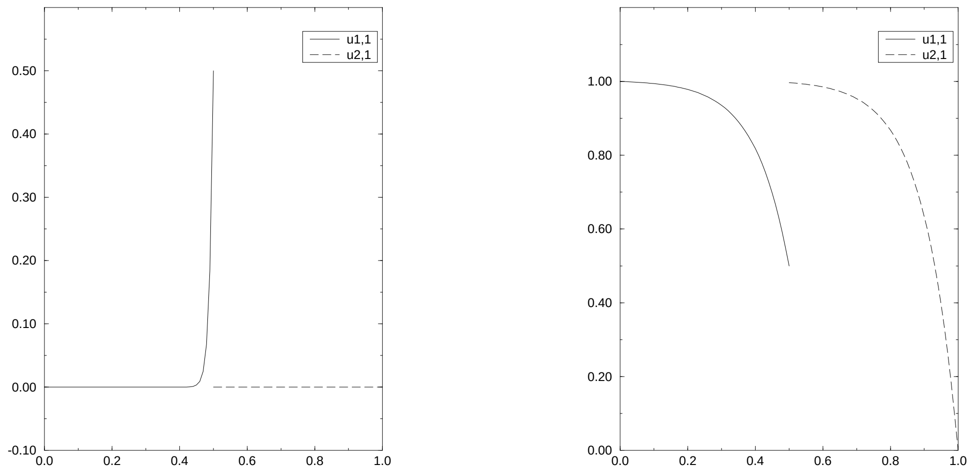

Figure 2.1. The smoothing of the artificial internal layer for $A = -\infty$, $B = \frac{1}{2}b$.

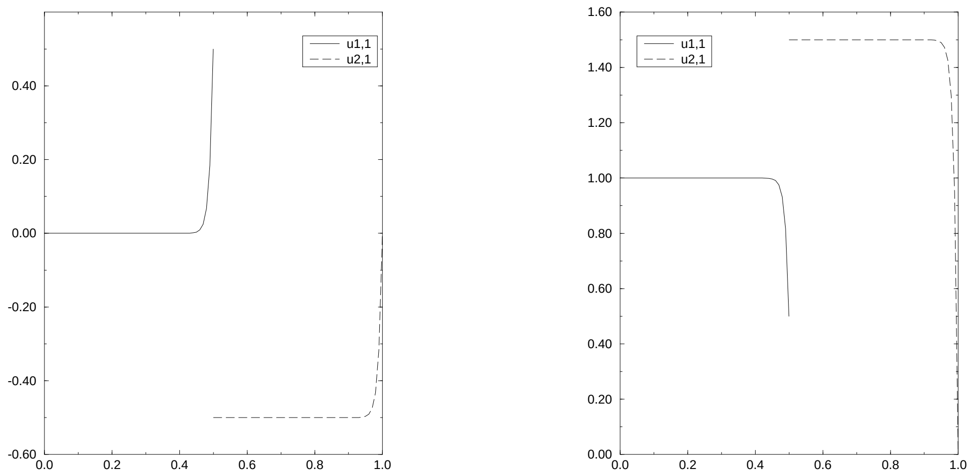

Figure 2.2. The smoothing of the artificial internal layer for $A = -\infty$, $B = 0$.

Notice also that, as in [11], our argument permits imposing the value of the normal derivative on an outflow boundary and similarly the value of the conormal derivative on an inflow boundary. Indeed, this corresponds for $b > 0$ to taking $A = -b/2 < 0$ (outflow for Ω_1) or $B = b/2 > 0$ (inflow for Ω_2), and for $b < 0$ to choosing $A = b/2 < 0$ (inflow for Ω_1) or $B = -b/2 > 0$ (outflow for Ω_2), and for all these choices coerciveness is guaranteed.

Finally, if the boundary has an inflow and an outflow part at the same time, we claim that it is not necessary to employ an adaptive strategy on the interface, but it is sufficient to impose a set of boundary conditions which assures coerciveness of the bilinear forms in both subdomains. In the next Sections 3 and 4 we are going to present two families of boundary value problems which enjoy these properties.

3. The γ-DR iterative scheme

We propose the following iteration-by-subdomain scheme for solving (1.1), which will be called *γ-Dirichlet/Robin* (γ-DR).

Define by Λ the trace space on Γ of $H_0^1(\Omega)$. It can be shown that this space coincides with the Sobolev space $H_{00}^{1/2}(\Gamma)$ (for the definition of this space, see, e.g., [13]).

The scheme reads: let λ^0 be given in Λ, for each $k \geq 0$ solve

$$
(3.1) \qquad \begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma \end{cases}
$$

$$
(3.2) \qquad \begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} - \left(\dfrac{1}{2} \mathbf{b} \cdot \mathbf{n} + \gamma \right) u_2^{k+1} & \\ \qquad = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2} \mathbf{b} \cdot \mathbf{n} + \gamma \right) u_1^{k+1} & \text{on } \Gamma \ , \end{cases}
$$

and set

$$
(3.3) \qquad \lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma \ ,
$$

where $\theta \neq 0$ is a relaxation parameter introduced to accelerate convergence.

In (3.2) $\gamma = \gamma(\mathbf{x})$ is a given function belonging to $L^\infty(\Gamma)$, satisfying $\gamma(\mathbf{x}) \geq 0$ for almost each $\mathbf{x} \in \Gamma$; the rate of convergence of the method is in principle dependent on the choice of this function. Compared with the scheme analized in Section 2, we are choosing here $A = -\infty$ and $B = \gamma$.

To ensure the solvability of problems (3.1) and (3.2), it is useful to consider their variational formulation. Let us define for $i = 1, 2$

$$
V_i := \{ v_i \in H^1(\Omega_i) \,|\, v_{i|\partial\Omega \cap \partial\Omega_i} = 0 \}
$$

and introduce the local bilinear forms

$$
(3.4) \qquad \begin{aligned} a_i^\sharp(w_i, v_i) : &= \int_{\Omega_i} \left[\varepsilon \nabla w_i \cdot \nabla v_i + \left(\frac{1}{2} \operatorname{div} \mathbf{b} + a_0 \right) w_i \, v_i \right] + \\ & \qquad + \frac{1}{2} \int_\Omega (v_i \, \mathbf{b} \cdot \nabla w_i - w_i \, \mathbf{b} \cdot \nabla v_i) \ . \end{aligned}
$$

Notice that, from (1.2), there exist constants $\beta_i^\sharp$ and $\alpha_i^\sharp$, $i = 1, 2$, such that

$$
(3.5) \qquad a_i^\sharp(w_i, v_i) \leq \beta_i^\sharp \, ||w_i||_{1,\Omega_i} \, ||v_i||_{1,\Omega_i} \quad \forall \ w_i, v_i \in V_i
$$

and

$$
(3.6) \qquad a_i^\sharp(v_i, v_i) \geq \alpha_i^\sharp \, ||v_i||_{1,\Omega_i}^2 \quad \forall \ v_i \in V_i \ .
$$

The iterative scheme (3.1)-(3.3) reads:

$$
(3.7) \qquad \begin{cases} \text{find } \ u_1^{k+1} \in V_1 \ : \\ a_1^\sharp(u_1^{k+1}, v_1) = \displaystyle\int_{\Omega_1} f v_1 \qquad \forall \ v_1 \in H_0^1(\Omega_1) \\ u_{1|\Gamma}^{k+1} = \lambda^k \end{cases}
$$

$$
(3.8) \qquad \begin{cases}
\text{find } \ u_2^{k+1} \in V_2 \ : \\
a_2^\sharp(u_2^{k+1}, v_2) = \displaystyle\int_{\Omega_2} f v_2 \qquad \forall\ v_2 \in H_0^1(\Omega_2) \\
a_2^\sharp(u_2^{k+1}, \mathcal{R}_2\mu) + \displaystyle\int_\Gamma \gamma u_{2|\Gamma}^{k+1}\mu = \int_{\Omega_2} f\mathcal{R}_2\mu + \int_{\Omega_1} f\mathcal{R}_1\mu \\
\qquad\qquad -a_1^\sharp(u_1^{k+1}, \mathcal{R}_1\mu) + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1}\mu \qquad \forall\ \mu \in \Lambda
\end{cases}
$$

and finally

$$
(3.9) \qquad \lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma \ ,
$$

where $\mathcal{R}_i$ denotes any extension operator from Λ to V_i.

Problem (3.8) can be rewritten in the equivalent form

$$
(3.10) \quad \begin{cases}
\text{find } \ u_2^{k+1} \in V_2 \ : \\
a_2^\sharp(u_2^{k+1}, v_2) + \displaystyle\int_\Gamma \gamma u_{2|\Gamma}^{k+1} v_{2|\Gamma} = \int_{\Omega_2} f v_2 + \int_{\Omega_1} f\mathcal{R}_1 v_{2|\Gamma} \\
\qquad\qquad -a_1^\sharp(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\ v_2 \in V_2 \ .
\end{cases}
$$

It must be noticed that problem (3.7) is a coercive problem in $H_0^1(\Omega_1)$, whereas problem (3.10) is coercive in V_2, for any $\gamma \geq 0$. Hence the iterative scheme is correctly defined, and, more important, enjoys the coerciveness properties which have been shown in Section 2 to lead to convergent schemes. We want also to underline that it is different from the ADN scheme proposed in [11], as the Dirichlet boundary condition is imposed on the whole interface Γ, no matter if it is an inflow or an outflow boundary. However, in the particular situation in which the flow has always the same direction on Γ, say $\mathbf{b}\cdot\mathbf{n} < 0$ on Γ, choosing $\gamma = -\frac{1}{2}\,\mathbf{b}\cdot\mathbf{n}$ we recover the ADN scheme.

We also propose a modified algorithm, which is somehow more complicated to implement, but enjoys better convergence properties. Setting $((\eta,\mu))_\Lambda$ the scalar product in the trace space $\Lambda = H_{00}^{1/2}(\Gamma)$, we solve instead of (3.10) the following problem

$$
(3.11) \ \begin{cases}
\text{find } \ u_2^{k+1} \in V_2 \ : \\
a_2^\sharp(u_2^{k+1}, v_2) + \gamma((u_{2|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda = \displaystyle\int_{\Omega_2} f v_2 + \int_{\Omega_1} f\mathcal{R}_1 v_{2|\Gamma} \\
\qquad\qquad -a_1^\sharp(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \gamma((u_{1|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda \qquad \forall\ v_2 \in V_2 \ ,
\end{cases}
$$

for a constant $\gamma \geq 0$.

For proving the convergence of this scheme, we need some preliminary results. First of all, for $i = 1, 2$ and for each $\eta \in \Lambda$, introduce the solution $E_i\eta \in V_i$ of the Dirichlet boundary value problem

$$
\begin{cases} L_\varepsilon(E_i\eta) = 0 & \text{in } \Omega_i \\ (E_i\eta)_{|\partial\Omega\cap\partial\Omega_i} = 0 & \\ (E_i\eta)_{|\Gamma} = \eta \ . & \end{cases} \tag{3.12}
$$

By well-known a-priori estimates for elliptic problems, the extension operator $E_i : \Lambda \to V_i$ is continuous, i.e., there exists $k_i > 0$ such that

$$
||E_i\eta||_{1,\Omega_i} \le k_i ||\eta||_\Lambda \quad \forall\ \eta \in \Lambda \ . \tag{3.13}
$$

Moreover, the trace inequality yields

$$
\tilde{k}_i ||\eta||_\Lambda \le ||E_i\eta||_{1,\Omega_i} \quad \forall\ \eta \in \Lambda \tag{3.14}
$$

for a suitable constant $\tilde{k}_i > 0$.

For each $\eta, \mu \in \Lambda$ define now the Steklov-Poincaré operators $S_i : \Lambda \to \Lambda'$ as

$$
\begin{aligned} \langle S_1\eta, \mu\rangle &= a_1^\sharp(E_1\eta, E_1\mu) - \gamma((\eta, \mu))_\Lambda \\ \langle S_2\eta, \mu\rangle &= a_2^\sharp(E_2\eta, E_2\mu) + \gamma((\eta, \mu))_\Lambda \ , \end{aligned} \tag{3.15}
$$

and set

$$
S = S_1 + S_2 \ .
$$

The operator S_1 turns out to be continuous as

$$
\langle S_1\eta, \mu\rangle \le (\beta_1^\sharp k_1^2 + \gamma)||\eta||_\Lambda ||\mu||_\Lambda \ , \tag{3.16}
$$

and moreover, for each $\gamma \ge 0$, S_2 is continuous and coercive, as

$$
\langle S_2\eta, \mu\rangle \le (\beta_2^\sharp k_2^2 + \gamma)||\eta||_\Lambda ||\mu||_\Lambda \tag{3.17}
$$

and

$$
\langle S_2\eta, \eta\rangle \ge (\alpha_2^\sharp \tilde{k}_2^2 + \gamma)||\eta||_\Lambda^2 \ . \tag{3.18}
$$

It is easily seen that the iteration operator in (3.7), (3.11), (3.9) is given by

$$
T_\theta := I - \theta S_2^{-1} S \tag{3.19}
$$

(see for instance [1], Section 5, for the same result in a different context). The proof of convergence is therefore reduced to showing that the operator T_θ is a contraction in Λ, with respect to a suitable norm.

We need the following abstract convergence theorem:

Theorem 3.1. *Let X be a (real) Hilbert space and X' its dual space, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X' and X. Let the linear continuous operator $S : X \to X'$ be split as $S = S_1 + S_2$. Suppose that*

1. S_1 is linear and continuous, i.e., there exists $\beta_1 > 0$ such that

$$\langle S_1 \eta, \mu \rangle \leq \beta_1 \, ||\eta||_X \, ||\mu||_X \quad \forall \, \eta, \mu \in X \ ;$$

2. S_2 is linear, continuous and coercive, i.e.

2.a there exists $\beta_2 > 0$ such that

$$\langle S_2 \eta, \mu \rangle \leq \beta_2 \, ||\eta||_X \, ||\mu||_X \quad \forall \, \eta, \mu \in X \ ;$$

2.b there exists $\alpha_2 > 0$ such that

$$\langle S_2 \eta, \eta \rangle \geq \alpha_2 \, ||\eta||_X^2 \quad \forall \, \eta \in X \ ;$$

3. there exists a constant $\kappa^* > 0$ such that

$$\langle S_2 \eta, S_2^{-1} S \eta \rangle + \langle S \eta, \eta \rangle \geq \kappa^* ||\eta||_X^2 \quad \forall \, \eta \in X \ .$$

Then for any given λ^0 in X the sequence

$$\lambda^{k+1} = \lambda^k - \theta S_2^{-1} S \lambda^k$$

converges to 0, provided that

$$0 < \theta < \frac{\kappa^* \alpha_2^2}{\beta_2 (\beta_1 + \beta_2)^2} \ .$$

Proof. We introduce the scalar product

$$(\eta, \mu)_{S_2} := \frac{1}{2} \left(\langle S_2 \eta, \mu \rangle + \langle S_2 \mu, \eta \rangle \right) \ ,$$

with the corresponding norm $||\eta||_{S_2} := \langle S_2 \eta, \eta \rangle^{1/2}$, which is equivalent to the norm $||\eta||_X$, i.e.,

$$\alpha_2 ||\eta||_X^2 \leq ||\eta||_{S_2}^2 \leq \beta_2 ||\eta||_X^2 \ .$$

We prove that the map $T_\theta : X \to X$ defined as

$$T_\theta \eta := \eta - \theta S_2^{-1} S \eta$$

is a contraction with respect to the norm $|| \cdot ||_{S_2}$. Assuming that $0 \leq \theta$, we have

$$\begin{aligned} ||T_\theta \eta||_{S_2}^2 &= ||\eta||_{S_2}^2 + \theta^2 \langle S \eta, S_2^{-1} S \eta \rangle - \theta (\langle S_2 \eta, S_2^{-1} S \eta \rangle + \langle S \eta, \eta \rangle) \\ &\leq ||\eta||_{S_2}^2 + \theta^2 \frac{(\beta_1 + \beta_2)^2}{\alpha_2} ||\eta||_X^2 - \theta \kappa^* ||\eta||_X^2 \ , \end{aligned}$$

and we obtain

$$||T_\theta \eta||^2_{S_2} \le K^*_\theta ||\eta||^2_{S_2} \ ,$$

where

$$K^*_\theta = 1 + \theta^2 \frac{(\beta_1+\beta_2)^2}{\alpha_2^2} - \theta \frac{\kappa^*}{\beta_2}$$

The thesis follows by imposing the condition $K^*_\theta < 1$. $\square$

We are in a position to prove

Theorem 3.2. *There exists* $\gamma^* \ge 0$ *such that for each* $\gamma \ge \gamma^*$ *and for each* $\lambda^0 \in \Lambda$ *the iterative scheme* (3.7), (3.11), (3.9) *is convergent in* Λ, *provided that the relaxation parameter* θ *is chosen in a suitable interval* $(0, \theta_\gamma)$.

Proof. From (3.16),(3.17) and (3.18) assumptions 1 and 2 of Theorem 3.1 are satisfied with

$$\beta_i = \beta_i^\sharp k_i^2 + \gamma \quad , \quad \alpha_2 = \alpha_2^\sharp \tilde{k}_2^2 + \gamma \ ,$$

$i = 1, 2$. We are going to prove that there exists $\gamma^* \ge 0$ such that for each $\gamma \ge \gamma^*$ assumption 3 of Theorem 3.1 is satisfied. We have

$$\begin{aligned}\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle &= 2\langle S\eta,\eta\rangle + \langle S_2\eta, S_2^{-1}S\eta\rangle - \langle S\eta,\eta\rangle \\ &\ge 2\langle S\eta,\eta\rangle - |\langle S_2\eta, S_2^{-1}S\eta\rangle - \langle S\eta,\eta\rangle| \ .\end{aligned}$$

Setting $\mu = S_2^{-1}S\eta$, one obtains

$$\begin{aligned}|\langle S_2\eta, S_2^{-1}S\eta\rangle - \langle S\eta,\eta\rangle| &= |a_2^\sharp(E_2\eta, E_2\mu) - a_2^\sharp(E_2\mu, E_2\eta)| \\ &= |\int_{\Omega_2} \mathbf{b}\cdot(E_2\mu\nabla E_2\eta - E_2\eta\nabla E_2\mu)| \\ &\le 2||\mathbf{b}||_{L^\infty(\Omega_2)}||E_2\eta||_{1,\Omega_2}||E_2\mu||_{1,\Omega_2} \\ &\le 2||\mathbf{b}||_{L^\infty(\Omega_2)}k_2^2||\eta||_\Lambda||S_2^{-1}S\eta||_\Lambda \ .\end{aligned}$$

From the definition of S and (3.5),(3.6), (3.13) and (3.14) we find

$$\langle S\eta,\mu\rangle \le (\beta_1^\sharp k_1^2 + \beta_2^\sharp k_2^2)\,||\eta||^2_\Lambda ||\mu||^2_\Lambda \ ,$$

$$\langle S\eta,\eta\rangle \ge (\alpha_1^\sharp \tilde{k}_1^2 + \alpha_2^\sharp \tilde{k}_2^2)\,||\eta||^2_\Lambda \ .$$

Therefore, setting $\beta := \beta_1^\sharp k_1^2 + \beta_2^\sharp k_2^2$ and $\alpha := \alpha_1^\sharp \tilde{k}_1^2 + \alpha_2^\sharp \tilde{k}_2^2$, we have

$$\begin{aligned}\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle &\ge 2\alpha||\eta||^2_\Lambda - 2||\mathbf{b}||_{L^\infty(\Omega_2)}k_2^2\frac{\beta}{\alpha_2}||\eta||^2_\Lambda \\ &= 2\left(\alpha - ||\mathbf{b}||_{L^\infty(\Omega_2)}k_2^2\frac{\beta}{\alpha_2}\right)||\eta||^2_\Lambda \ .\end{aligned}$$

The assumption 3 is satisfied provided that

$$\kappa^* := 2\left(\alpha - ||\mathbf{b}||_{L^\infty(\Omega_2)}k_2^2\frac{\beta}{\alpha_2}\right) > 0 \ ,$$

i.e.,

$$\alpha_2 > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha} \ .$$

Recalling the definition of α_2 it is sufficient to take

$$\gamma \geq \gamma^* := \begin{cases} 0 & \text{for } \alpha_2^\sharp \tilde{k}_2^2 > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha} \\ > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha} - \alpha_2^\sharp \tilde{k}_2^2 & \text{for } \alpha_2^\sharp \tilde{k}_2^2 \leq ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha} \end{cases} ,$$

and finally apply Theorem 3.1. □

It is worthwhile to notice that the rate of convergence of the iterative scheme (3.7), (3.11), (3.9) only depends on the parameters $\beta_i^\sharp$, $\alpha_i^\sharp$ in (3.5) and (3.6), k_i, $\tilde{k}_i$ in (3.13) and (3.14), $i = 1, 2$. When considering a finite dimensional approximation, all these constants, except k_i, are independent of the total number of degrees of freedom. Therefore, the iterative scheme furnishes an optimal preconditioner, provided that we can find an uniform bound for k_i. In other words, it is necessary to prove the uniform extension result

$$(3.20) \qquad ||E_{i,h}\eta_h||_{1,\Omega_i} \leq k_i ||\eta_h||_\Lambda \quad \forall\, \eta_h \in \Lambda_h \ ,$$

where Λ_h is the discrete approximation of the trace space Λ, and, for each $\eta \in \Lambda$, $E_{i,h}\eta$ is the finite dimensional counterpart of $E_i\eta$ introduced in (3.12).

This result is well-known, e.g., for piecewise-polynomial finite elements defined on a regular family of triangulations $\mathcal{T}_h$ of Ω, which induces a quasi-uniform family of triangulations on Γ (see, for instance, [5], [4], [15]).

Remark 3.3. In the finite dimensional case, the convergence of the iterative scheme (3.7)-(3.9) can be proven by a similar argument. In fact, for discrete functions all the norms are equivalent, hence there exists a constant $\kappa_h > 0$ such that

$$(3.21) \qquad \kappa_h ||\eta_h||_\Lambda^2 \leq ||\eta_h||_{0,\Gamma}^2 \quad \forall\, \eta_h \in \Lambda_h \ .$$

By using this estimate, we only have to substitute the constant $\alpha_2 := \alpha_2^\sharp \tilde{k}_2^2 + \gamma$ in (3.18) with

$$\alpha_{2,h} := \alpha_2^\sharp \tilde{k}_2^2 + \gamma \kappa_h \ ,$$

and convergence is achieved for $\inf_\Omega \gamma \geq \gamma_h^* := \gamma^*/\kappa_h$.

Therefore, in this case we are not in a condition to prove that the iterative procedure introduces an optimal preconditioner. However, the numerical results shows that the rate of convergence is in fact independent of h (see Section 6). □

Remark 3.4. Though the convergence result in Theorem 3.2 holds only for γ sufficiently large, numerical evidence shows that the γ-DR iterative scheme indeed converges for any $\gamma \geq 0$, in particular for $\gamma = 0$. In Section 6 we are really going to apply only the 0-DR method, to all the numerical test cases.$\square$

4. The γ-RR iterative scheme

In this Section we present and analyze another iteration-by-subdomain procedure, which will be called *γ-Robin/Robin* (γ-RR). It reads as follows: given λ^0 in $L^2(\Gamma)$, for each $k \geq 0$ solve

$$(4.1)\qquad \begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} - \gamma\right) u_1^{k+1} = \lambda^k & \text{on } \Gamma \end{cases}$$

and

$$(4.2)\qquad \begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} - \left(\dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_2^{k+1} & \\ \qquad = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} & \text{on } \Gamma \;, \end{cases}$$

where

$$(4.3)\qquad \lambda^{k+1} := \varepsilon \frac{\partial u_2^{k+1}}{\partial n} - \left(\frac{1}{2}\mathbf{b}\cdot\mathbf{n} - \gamma\right) u_2^{k+1} \qquad \text{on } \Gamma \;,$$

$\gamma = \gamma(\mathbf{x})$ being a given function in $L^\infty(\Gamma)$ satisfying $\gamma(\mathbf{x}) \geq \hat{\gamma} > 0$ for almost each $\mathbf{x} \in \Gamma$. With respect to the method introduced in Section 2, we are setting here $A = -\gamma$ and $B = \gamma$.

Noticing that

$$(4.4)\qquad \begin{aligned} \lambda^{k+1} &= \varepsilon \frac{\partial u_1^{k+1}}{\partial n} - \left(\frac{1}{2}\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} + 2\gamma u_2^{k+1} \\ &= \lambda^k + 2\gamma(u_2^{k+1} - u_1^{k+1}) \;, \end{aligned}$$

we can rewrite the scheme above in the variational form

$$(4.5)\qquad \begin{aligned} \text{find } u_1^{k+1} \in V_1 \;:\; a_1^\sharp(u_1^{k+1}, v_1) &+ \int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{1|\Gamma} \\ &= \int_{\Omega_1} f v_1 + \int_\Gamma \lambda^k v_{1|\Gamma} \qquad \forall\, v_1 \in V_1 \;, \end{aligned}$$

then

$$
\text{find } u_2^{k+1} \in V_2 \ : \ a_2^\sharp(u_2^{k+1}, v_2) + \int_\Gamma \gamma u_{2|\Gamma}^{k+1} v_{2|\Gamma}
$$

$$
(4.6) \qquad = \int_{\Omega_2} f v_2 + \int_{\Omega_1} f \mathcal{R}_1 v_{2|\Gamma} - a_1^\sharp(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma})
$$

$$
+ \int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2 \ ,
$$

and finally

$$
(4.7) \qquad \lambda^{k+1} = \lambda^k + 2\gamma(u_{2|\Gamma}^{k+1} - u_{1|\Gamma}^{k+1}) \qquad \text{on } \Gamma \ ,
$$

where notation is as in Section 3. Due to the assumption $\gamma \in L^\infty(\Gamma)$, we have that $\lambda^{k+1} \in L^2(\Gamma)$.

Let us underline that, also in the present case, the bilinear forms which are used in the iterative scheme, i.e.,

$$
a_i^\sharp(w_i, v_i) + \int_\Gamma \gamma w_i v_i \ , \ i = 1, 2 \ ,
$$

are coercive in V_i, for each $\gamma \geq 0$.

We obtain the following convergence theorem which is inspired by the results of [14], [16]:

Theorem 4.1. *Assume either that Ω is a Lipschitz polygonal domain or that $\partial\Omega \in C^2$. Suppose moreover that $\mathbf{b}_{|\Gamma} \in (L^\infty(\Gamma))^d$. For each $\lambda^0 \in L^2(\Gamma)$ and for each $i = 1, 2$, the sequences u_i^k converge in $H^1(\Omega_i)$ to the restriction $u_{|\Omega_i}$ of the solution u of* (1.4).

Proof. Set $e_i^k := u_i^k - u_{|\Omega_i}$ for each $k \geq 0$. The exact solution u clearly satisfies

$$
a_1^\sharp(u_{|\Omega_1}, v_1) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma} = \int_{\Omega_1} f v_1 + \int_{\Omega_2} f \mathcal{R}_2 v_{1|\Gamma}
$$

$$
- a_2^\sharp(u_{|\Omega_2}, \mathcal{R}_2 v_{1|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma} \qquad \forall\, v_1 \in V_1 \ ,
$$

and

$$
a_2^\sharp(u_{|\Omega_2}, v_2) + \int_\Gamma \gamma u_{|\Gamma} v_{2|\Gamma} = \int_{\Omega_2} f v_2 + \int_{\Omega_1} f \mathcal{R}_1 v_{2|\Gamma}
$$

$$
- a_1^\sharp(u_{|\Omega_1}, \mathcal{R}_1 v_{2|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2 \ .
$$

From well-known regularity results for elliptic equations (see, e.g., [13], [10]), the solution u belongs to $H^{3/2+\delta}(\Omega)$ for a suitable $\delta > 0$, and consequently $\frac{\partial u}{\partial n} \in L^2(\Gamma)$. Therefore we can also write

$$
\int_{\Omega_2} f \mathcal{R}_2 v_{1|\Gamma} - a_2^\sharp(u_{|\Omega_2}, \mathcal{R}_2 v_{1|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma}
$$

$$
= \int_\Gamma \left(\varepsilon \frac{\partial u}{\partial n} - \frac{1}{2} \mathbf{b} \cdot \mathbf{n}\, u + \gamma\, u \right)_{|\Gamma} v_{1|\Gamma} \ .
$$

Setting now

$$\omega^k := \lambda^k - \left(\varepsilon \frac{\partial u}{\partial n} - \frac{1}{2} \mathbf{b} \cdot \mathbf{n} u + \gamma u \right)_{|\Gamma} ,$$

the error equations can be written as

$$a_1^\sharp(e_1^{k+1}, v_1) + \int_\Gamma \gamma e_{1|\Gamma}^{k+1} v_{1|\Gamma} = \int_\Gamma \omega^k v_{1|\Gamma} \qquad \forall\, v_1 \in V_1 , \tag{4.8}$$

and

$$\begin{aligned} & a_2^\sharp(e_2^{k+1}, v_2) + \int_\Gamma \gamma e_{2|\Gamma}^{k+1} v_{2|\Gamma} \\ & \qquad = -a_1^\sharp(e_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \int_\Gamma \gamma e_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2 , \end{aligned} \tag{4.9}$$

where

$$\omega^{k+1} = \omega^k + 2\gamma (e_{2|\Gamma}^{k+1} - e_{1|\Gamma}^{k+1}) . \tag{4.10}$$

Taking $v_1 = e_1^{k+1}$ in (4.8) and $v_2 = e_2^{k+1}$ in (4.9), we have

$$a_1^\sharp(e_1^{k+1}, e_1^{k+1}) = \int_\Gamma (\omega^k - \gamma e_{1|\Gamma}^{k+1}) e_{1|\Gamma}^{k+1} \tag{4.11}$$

and

$$a_2^\sharp(e_2^{k+1}, e_2^{k+1}) = -a_1^\sharp(e_1^{k+1}, \mathcal{R}_1 e_{2|\Gamma}^{k+1}) + \int_\Gamma \gamma (e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) e_{2|\Gamma}^{k+1} . \tag{4.12}$$

Choosing $v_1 = \mathcal{R}_1 e_{2|\Gamma}^{k+1}$ in (4.8), we also obtain

$$a_1^\sharp(e_1^{k+1}, \mathcal{R}_1 e_{2|\Gamma}^{k+1}) = \int_\Gamma (\omega^k - \gamma e_{1|\Gamma}^{k+1}) e_{2|\Gamma}^{k+1} , \tag{4.13}$$

and inserting this result in (4.12) we have

$$a_2^\sharp(e_2^{k+1}, e_2^{k+1}) = \int_\Gamma (2\gamma e_{1|\Gamma}^{k+1} - \gamma e_{2|\Gamma}^{k+1} - \omega^k) e_{2|\Gamma}^{k+1} . \tag{4.14}$$

Adding (4.11) and (4.14) we find

$$\begin{aligned} & a_1^\sharp(e_1^{k+1}, e_1^{k+1}) + a_2^\sharp(e_2^{k+1}, e_2^{k+1}) \\ & \qquad = \int_\Gamma \frac{1}{\gamma} [\gamma \omega^k (e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) - \gamma^2 (e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1})^2] \\ & \qquad = \int_\Gamma -\frac{1}{4\gamma} [2\gamma (e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) - \omega^k]^2 + \int_\Gamma \frac{1}{4\gamma} (\omega^k)^2 . \end{aligned}$$

Recalling (4.10), we finally obtain

$$a_1^\sharp(e_1^{k+1}, e_1^{k+1}) + a_2^\sharp(e_2^{k+1}, e_2^{k+1}) + \int_\Gamma \frac{1}{4\gamma}(\omega^{k+1})^2 = \int_\Gamma \frac{1}{4\gamma}(\omega^k)^2 \ . \tag{4.15}$$

Adding now over k from 0 to $M-1$, it follows

$$\begin{aligned} \sum_{k=1}^{M}[a_1^\sharp(e_1^k, e_1^k) + a_2^\sharp(e_2^k, e_2^k)] + \int_\Gamma \frac{1}{4\gamma}(\omega^M)^2 \\ = \int_\Gamma \frac{1}{4\gamma}(\omega^0)^2 \ , \end{aligned} \tag{4.16}$$

hence the series

$$\sum_{k=1}^{\infty}[a_1^\sharp(e_1^k, e_1^k) + a_2^\sharp(e_2^k, e_2^k)]$$

is convergent, and, as a consequence of the coerciveness of $a_i^\sharp(\cdot,\cdot)$ in V_i, e_i^k converge to 0 in $H^1(\Omega_i)$, $i = 1, 2$. □

Remark 4.2. The same result holds true when considering the corresponding discrete scheme, which can be obtained from (4.5)-(4.7) by substituting V_i, Λ and $\mathcal{R}_i$ with suitable finite dimensional approximations $V_{i,h}$, Λ_h and $\mathcal{R}_{i,h}$, respectively.

In fact, for each $\mu_h \in \Lambda_h$ we can write

$$\begin{aligned} \int_{\Omega_2} f\mathcal{R}_{2,h}\mu_h - a_2^\sharp(u_{h|\Omega_2}, \mathcal{R}_{2,h}\mu_h) + \int_\Gamma \gamma u_{h|\Gamma}\mu_h \\ = \int_\Gamma g_h \mu_h \end{aligned}$$

for a suitable $g_h \in \Lambda_h$. Hence, only assuming that Ω is a Lipschitz polygonal domain, for each initial guess $\lambda_h^0 \in \Lambda_h$ one obtains convergence as in Theorem 4.1.

Though we have no information on the rate of convergence, which, in principle, can depend on h, the numerical results in Section 6 show that this is not the case, and the number of subdomain iterations is independent of the mesh size, for suitable choices of the parameter $\gamma = \gamma_h$. □

Remark 4.3. It is worthwhile to notice that the γ-RR method generalizes several other ones proposed recently. For instance, we have

$$\gamma = \begin{cases} \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}| & \text{unrelaxed ARN method in [11]} \\ \frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4a_0\varepsilon} & \text{[16]} \\ \frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4\kappa\varepsilon}\ ,\ \kappa > 0 & \text{[2]} \end{cases} .$$

This is not the case for the AR$_\beta$N method in [11]. □

Remark 4.4. A possible strategy for choosing the parameter γ is the following one: minimize with respect to γ the upper bound

$$\int_\Gamma \frac{1}{4\gamma}(\omega^0)^2$$

in (4.16). Since

$$\omega^0 = \lambda^0 - \left(\varepsilon\frac{\partial u}{\partial n} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}u\right)_{|\Gamma} - \gamma u_{|\Gamma} \ ,$$

the minimum is attained for

$$\gamma = \sqrt{\frac{F}{G}} \ ,$$

where

$$F := \frac{1}{4}\int_\Gamma \left(\lambda^0 - \varepsilon\frac{\partial u}{\partial n} + \frac{1}{2}\mathbf{b}\cdot\mathbf{n}u\right)^2 \quad , \quad G := \frac{1}{4}\int_\Gamma u^2 \ .$$

Since the value of the exact solution u is not available, we propose to use along the iterations

$$\gamma_k := \sqrt{\frac{F_k}{G_k}} \quad , \quad k \geq 1 \ , \tag{4.17}$$

where

$$F_k := \frac{1}{4}\int_\Gamma \left(\lambda^0 - \varepsilon\frac{\partial u_2^k}{\partial n} + \frac{1}{2}\mathbf{b}\cdot\mathbf{n}u_2^k\right)^2 \quad , \quad G_k := \frac{1}{4}\int_\Gamma (u_2^k)^2 \ .$$

Again, in this situation we have not a convergence proof, but some numerical results show that this strategy works well enough (see Section 6). □

5. The γ-DR and γ-RR iterative schemes for systems

The iterative schemes introduced in Sections 3 and 4 can be used also when considering advection-diffusion systems, like

$$\begin{cases} -\varepsilon\Delta\mathbf{u} + \displaystyle\sum_{j=1}^{d} D_j(B^{(j)}\mathbf{u}) + A_0\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0} \ , \end{cases} \tag{5.1}$$

where $B^{(j)}$, $j = 1, ..., d$, A_0 are $q \times q$ *symmetric* matrices. We assume that the coefficients of $B^{(j)}$ and A_0 belong to $L^\infty(\Omega)$, and that the coefficients of $\sum_j D_j B^{(j)}$ belong to $L^\infty(\Omega)$. Moreover, we require that the matrix

$$M(\mathbf{x}) := \frac{1}{2}\sum_{j=1}^{d} D_j B^{(j)}(\mathbf{x}) + A_0(\mathbf{x}) \tag{5.2}$$

is positive semi-definite for almost each $\mathbf{x} \in \Omega$, which corresponds to the coerciveness assumption (1.2).

We can introduce the associated bilinear form

$$a^\sharp(\mathbf{w},\mathbf{v}) := \int_\Omega \left[\varepsilon\nabla\mathbf{w}\cdot\nabla\mathbf{v} + (M\,\mathbf{w})\cdot\mathbf{v}\right] + \frac{1}{2}\int_\Omega \sum_{j=1}^{d}[(B^{(j)}\mathbf{v})\cdot D_j\mathbf{w} - (B^{(j)}D_j\mathbf{v})\cdot\mathbf{w}]\ , \tag{5.3}$$

which can be used to rewrite the Dirichlet boundary value problem (5.1) in the variational form

$$\mathbf{u} \in (H^1_0(\Omega))^d\ :\ a^\sharp(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\ \mathbf{v} \in (H^1_0(\Omega))^d\ . \tag{5.4}$$

Let $\gamma = \gamma(\mathbf{x})$ be a $q\times q$ matrix with coefficients in $L^\infty(\Gamma)$ and positive semi-definite for almost each $\mathbf{x}\in\Gamma$. The γ-DR scheme for this problem reads:

$$\begin{cases} \text{find}\ \ \mathbf{u}_1^{k+1} \in (V_1)^q\ \ : \\ a_1^\sharp(\mathbf{u}_1^{k+1},\mathbf{v}_1) = \displaystyle\int_{\Omega_1} \mathbf{f}\cdot\mathbf{v}_1 \qquad \forall\ \mathbf{v}_1 \in (H^1_0(\Omega_1))^q \\ \mathbf{u}_{1|\Gamma}^{k+1} = \boldsymbol{\lambda}^k \end{cases} \tag{5.5}$$

$$\begin{cases} \text{find}\ \ \mathbf{u}_2^{k+1} \in (V_2)^q\ \ : \\ a_2^\sharp(\mathbf{u}_2^{k+1},\mathbf{v}_2) = \displaystyle\int_{\Omega_2} \mathbf{f}\cdot\mathbf{v}_2 \qquad \forall\ \mathbf{v}_2 \in (H^1_0(\Omega_2))^q \\ a_2^\sharp(\mathbf{u}_2^{k+1},\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + \displaystyle\int_\Gamma(\gamma\mathbf{u}_{2|\Gamma}^{k+1})\cdot\boldsymbol{\mu} = \int_{\Omega_2}\mathbf{f}\cdot\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu} + \int_{\Omega_1}\mathbf{f}\cdot\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu} \\ \qquad\qquad -a_1^\sharp(\mathbf{u}_1^{k+1},\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) + \displaystyle\int_\Gamma(\gamma\mathbf{u}_{1|\Gamma}^{k+1})\cdot\boldsymbol{\mu} \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^q \end{cases} \tag{5.6}$$

and finally

$$\boldsymbol{\lambda}^{k+1} := \theta\mathbf{u}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k \qquad \text{on } \Gamma\ , \tag{5.7}$$

with obvious meaning of notation.

On the other hand, choosing a $q\times q$ matrix $\gamma = \gamma(\mathbf{x})$ which is uniformly positive definite in Γ, the γ-RR scheme reads:

$$\text{find}\ \ \mathbf{u}_1^{k+1} \in (V_1)^q\ :\ a_1^\sharp(\mathbf{u}_1^{k+1},\mathbf{v}_1) + \int_\Gamma(\gamma\mathbf{u}_{1|\Gamma}^{k+1})\cdot\mathbf{v}_{1|\Gamma} = \int_{\Omega_1}\mathbf{f}\cdot\mathbf{v}_1 + \int_\Gamma \boldsymbol{\lambda}^k\cdot\mathbf{v}_{1|\Gamma} \qquad \forall\ \mathbf{v}_1 \in (V_1)^q\ , \tag{5.8}$$

then

$$
\text{(5.9)} \quad \begin{aligned}
\text{find } \mathbf{u}_2^{k+1} \in (V_2)^q \ : \ & a_2^\sharp(\mathbf{u}_2^{k+1}, \mathbf{v}_2) + \int_\Gamma (\gamma \mathbf{u}_{2|\Gamma}^{k+1}) \cdot \mathbf{v}_{2|\Gamma} \\
& = \int_{\Omega_2} \mathbf{f} \cdot \mathbf{v}_2 + \int_{\Omega_1} \mathbf{f} \cdot \boldsymbol{\mathcal{R}}_1 \mathbf{v}_{2|\Gamma} - a_1^\sharp(\mathbf{u}_1^{k+1}, \boldsymbol{\mathcal{R}}_1 \mathbf{v}_{2|\Gamma}) \\
& \quad + \int_\Gamma (\gamma \mathbf{u}_{1|\Gamma}^{k+1}) \cdot \mathbf{v}_{2|\Gamma} \qquad \forall\, \mathbf{v}_2 \in (V_2)^q \ ,
\end{aligned}
$$

and finally

$$
\text{(5.10)} \qquad \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + 2\gamma(\mathbf{u}_{2|\Gamma}^{k+1} - \mathbf{u}_{1|\Gamma}^{k+1}) \qquad \text{on } \Gamma \ .
$$

The convergence of both these iterative schemes can be shown as in Sections 3 and 4. More precisely, the γ-DR scheme is proven to converge provided that the matrix γ satisfies

$$
(\gamma(\mathbf{x})\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \gamma^* \boldsymbol{\xi} \cdot \boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathbf{R}^q \ , \text{ for almost each } \mathbf{x} \in \Gamma \ ,
$$

for a suitable $\gamma^* \geq 0$.

The γ-RR method converges provided that the matrix γ is diagonal and each entry γ_{ss}, $s = 1, ..., q$, satisfies

$$
\gamma_{ss}(\mathbf{x}) \geq \hat{\gamma} > 0 \qquad \text{for almost each } \mathbf{x} \in \Gamma \ .
$$

6. Numerical results

In this Section we present some numerical results, for different suitable test problems, obtained applying the γ-DR and γ-RR methods introduced before. Indeed, we are going to use the 0-DR method (namely γ-DR with $\gamma = 0$), which turns out to converge even if the theoretical results in general would require γ large enough, thus avoiding to propose a strategy for the choice of the parameter γ.

We implemented the schemes of Sections 3 and 4 on a cluster of an IBM RS/6000 workstations connected by Ethernet. The algorithms for the domain decomposition methods are parallelized using a Master/Slave paradigm in the PVM configuration.

When the advection is dominant, it is well-known that the pure Galerkin method for piecewise-polynomial finite elements is instable. Therefore we have employed the GALS stabilization method, which consists in substituting the bilinear form $a^\sharp(\cdot,\cdot)$ by

$$
a_h^\sharp(w_h, v_h) := a^\sharp(w_h, v_h) + \sum_{K \in \mathcal{T}_h} \tau_K (L_\varepsilon w_h, L_\varepsilon v_h)_K \ ,
$$

where $\mathcal{T}_h$ is the family of triangulations defined in Ω, $(\cdot,\cdot)_K$ denotes the $L^2(K)$-scalar product, and τ_K is a positive parameter which has to be chosen in a suitable way (see [12]). The right hand side $(f, v_h)_\Omega$ has to be changed correspondingly as

$$
\mathcal{F}_h(v_h) := (f, v_h)_\Omega + \sum_{K \in \mathcal{T}_h} \tau_K (f, L_\varepsilon v_h)_K \ .
$$

The iterative method used to solve the algebraic problems is CGSTAB with ILU preconditioner. The iterations of the CGSTAB method have been stopped when the relative error between two subsequent iterates is less than 10^{-11}, and the iterations over the subdomains when the relative $L^\infty(\Gamma)$-norm of the difference between two subsequent iterates is less then 10^{-10}, i.e., when

$$\frac{\|u_i^{k+1} - u_i^k\|_{L^\infty(\Gamma)}}{\|u_i^k\|_{L^\infty(\Gamma)}} \leq 10^{-10} \quad , \quad i = 1, 2\ . \tag{6.1}$$

6.1 First test case

We consider a test solution belonging to the space of the trial functions, which in our case are the piecewise-linear polynomials. We make such a simple choice to show the main features of the DD algorithms, as a test solution $u \in V_h$ avoids any approximation error and shows in an explicit way the algorithm behaviour with respect to the parameters.

We consider the problem $-\varepsilon\Delta u + \mathbf{b} \cdot \nabla u = f$, with $\mathbf{b} = (1, 1)$, $u(x, y) = x + 5y$ and f and the boundary conditions computed accordingly. The computational domain is $\Omega = (0, 1) \times (0, 1)$, which has been split in two rectangular subdomains Ω_1 and Ω_2.

We have applied to this problem the ADN schemes, the 0-DR scheme and the γ-RR scheme (with the value of γ obtained using formula (4.17), which in this case turns out to be nearly optimal).

We have used a mesh having 21×21 points in each subdomain. When implementing the ADN method, for each ε we have chosen the optimal value of θ reported in [17]. In general this value is rather sensitive to ε, and, for the example at hand, ranges between 0.5 and 0.8. Instead, for the 0-DR method we have observed that the optimal θ is equal to 0.5 for any choice of ε, provided that the ratio between the values of the mesh size in the two subdomains is equal to one, otherwise the optimal value of θ is not far from 0.5, as shown in Table 6.1. We also notice that it is not straightforward to find the optimal parameter θ for the γ-DR algorithm, $\gamma \neq 0$, and for a generic choice of θ we have verified that its convergence is often rather slow.

position of Γ	optimal θ	number of iterations
$x_\Gamma = 0.25$	0.54	12
$x_\Gamma = 0.50$	0.5	6
$x_\Gamma = 0.75$	0.44	14

Table 6.1. Optimal values of θ for the 0-DR method.

In the one-dimensional case considered in Section 2 the γ-DR scheme corresponds to the choice $A = -\infty$ and $B = \gamma$. It is worthwhile to notice that, in the limit $\varepsilon \to 0^+$, in that case the best choice of the parameter θ is the one for which

$$1 - \theta[1 - \rho_0(-\infty, \gamma)] = 0$$

(see (2.7), (2.9)), namely

$$\theta_{\rm opt} = \frac{1}{2} + \frac{\gamma}{|b|} \ .$$

This strengthens the conviction that the choice $\theta = 0.5$ for the multi-dimensional 0-DR scheme is likely close to the optimal one.

To make a comparison between the ADN, 0-DR and γ-RR methods, we show in Fig. 6.2 the number of iterations needed to achieve convergence, in the case the two subdomains of the same size, each one having 21×21 uniformly spaced grid points.

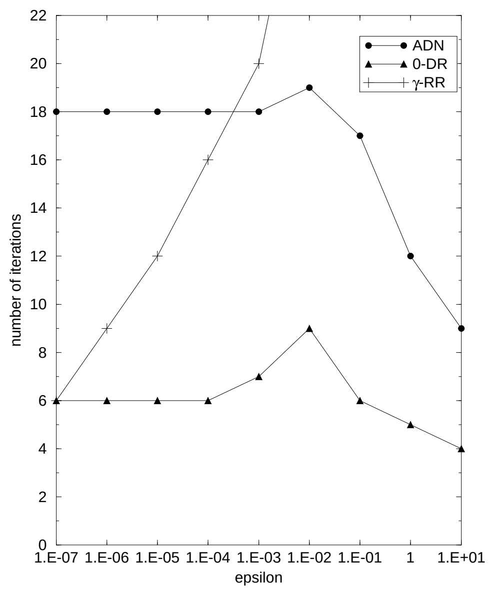

Figure 6.2. Number of iterations for the ADN, 0-DR and γ-RR methods.

It is worthwhile to notice that the 0-DR scheme performs better than the ADN one. In fact, the number of iterations needed is lower, without needing to modify the value of θ with respect to ε.

In Tables 6.3 and 6.4 it is shown that the rate of convergence of the 0-DR and γ-RR methods is essentially independent of the number of degrees of freedom. Moreover, the number of iterations of the 0-DR scheme depends very mildly on the value of ε. In this examples, we are splitting the domain Ω in two parts of the same size, having various meshes, with the same number of nodes in the direction x in Ω_1 and Ω_2, and always 21 nodes in the direction y.

nodes $\setminus$ ε	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$(21 \times 21) + (21 \times 21)$	4	5	6	9	7	6	6	6	6
$(31 \times 21) + (31 \times 21)$	4	4	6	9	7	6	6	6	6
$(41 \times 21) + (41 \times 21)$	4	4	5	7	7	7	6	6	6
$(51 \times 21) + (51 \times 21)$	4	4	5	7	8	7	6	6	7

Table 6.3. Number of iterations of the 0-DR method.

nodes \ ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$(21 \times 21) + (21 \times 21)$	20	16	12	9	6
$(31 \times 21) + (31 \times 21)$	20	16	12	9	6
$(41 \times 21) + (41 \times 21)$	20	16	12	9	6
$(51 \times 21) + (51 \times 21)$	20	16	12	9	6

Table 6.4. Number of iterations of the γ-RR method.

We finally notice that, for the problem at hand, the unrelaxed ARN method has the same behaviour of the γ-RR method (with the value of γ given by (4.17), which in this case is $\gamma \simeq 0.5$). The same happens for the choice of γ proposed in [16], and also for the one in [2] (when $\kappa\varepsilon$ is small enough).

6.2. The thermal boundary layer problem

This problem reads:

$$-\varepsilon\Delta u + 2yu_x = 0 \qquad \text{on } \Omega = (0,1) \times (0, 0.5)$$

with boundary conditions described in Fig. 6.5.

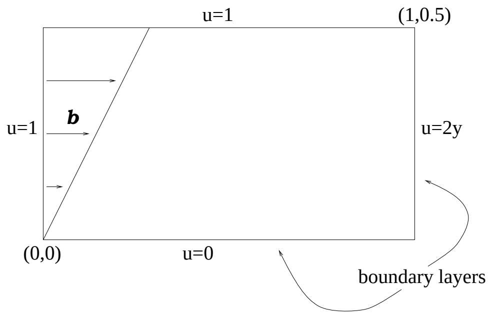

Figure 6.5. Boundary conditions for the thermal boundary layer problem.

The solution presents two zones of large gradient near the boundaries

$$\{y = 0 \text{ and } 0 \le x \le 1\}\ , \{x = 1 \text{ and } 0 \le y \le 0.5\}\ .$$

In Fig. 6.6 the convergence histories of the ADN, ARN, 0-DR and γ-RR methods are plotted for $\varepsilon = 10^{-4}$. We have divided Ω in two subdomains:

$\Omega_1 := (0, 0.7) \times (0, 0.5)$ with 21×41 uniformly spaced grid points

$\Omega_2 := (0.7, 1) \times (0, 0.5)$ with 41×41 uniformly spaced grid points.

The value of the relaxation parameter is $\theta = 0.91$ for the ADN scheme, $\theta = 0.42$ for the 0-DR scheme (due to the different mesh parameters in Ω_1 and Ω_2), and $\theta = 1$ for the ARN scheme (unrelaxed ARN scheme). For the γ-RR scheme the value of γ, which is obtained using formula (4.17), is approximately 0.13 for all the computations.

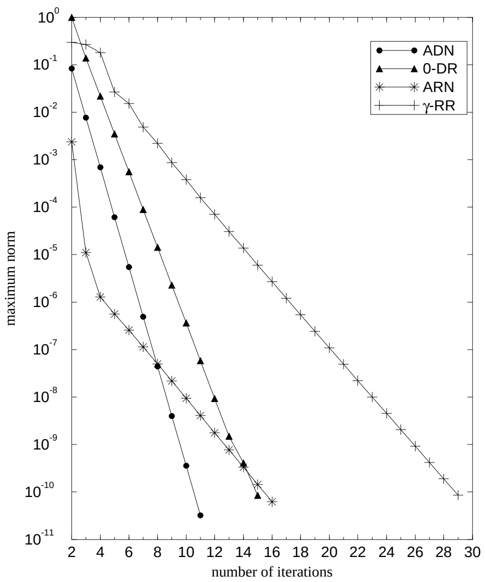

Figure 6.6. Convergence histories for $\varepsilon = 10^{-4}$.

Also in this example, the choice of the optimal relaxation parameter θ for the 0-DR scheme is rather easy, as it is exactly 0.5 when $h_{\Omega_1} = h_{\Omega_2}$, and close enough to that value in several other cases, as shown in Table 6.7. In this Table, the number of nodes in Ω_1 and Ω_2 is always 21×21.

position of Γ	optimal θ	number of iterations
$x_\Gamma = 0.25$	0.52	10
$x_\Gamma = 0.50$	0.5	7
$x_\Gamma = 0.75$	0.48	10

Table 6.7. Optimal values of θ for the 0-DR method.

Again, the rate of convergence turns out to be essentially independent of the number of degrees of freedom and of ε, for both the 0-DR and the γ-RR methods (see Tables 6.8 and 6.9). The choice of the parameter $\gamma = \gamma_h$ for the γ-RR method is now different from the one indicated in (4.17), and has been determined running the program a few times, looking for the "best" rate of convergence. In these last computations, we have divided the domain Ω in $\Omega_1 := (0, 0.75) \times (0, 0.5)$ and $\Omega_2 := (0.75, 1) \times (0, 0.5)$.

nodes \ ε	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$(31 \times 21) + (11 \times 21)$	10	10	8	10	8	7	6	6	6
$(46 \times 21) + (16 \times 21)$	10	10	8	10	8	7	6	6	6
$(61 \times 21) + (21 \times 21)$	10	10	8	10	8	7	7	8	9
$(76 \times 21) + (26 \times 21)$	10	10	8	10	8	8	11	7	6

Table 6.8. Number of iterations of the 0-DR method.

nodes \ ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$(31 \times 21) + (11 \times 21)$	30	26	25	26	25
$(46 \times 21) + (16 \times 21)$	32	25	25	26	25
$(61 \times 21) + (21 \times 21)$	32	27	26	26	26
$(76 \times 21) + (26 \times 21)$	31	27	26	27	26

Table 6.9. Number of iterations of the γ-RR method.

We notice that, for this thermal boundary layer problem, the choice of the parameter $\gamma = \frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4\kappa\varepsilon}$ proposed in [2] for the γ-RR method is more efficient, at least for small ε. In fact, choosing $\varepsilon = 10^{-6}$, κ ranging between 10^{-2} and 10^2, and the number of nodes as in Table 6.9, the number of iterations needed to achieve convergence is always equal to 8.

6.3. An example with $\Gamma_0 \neq \emptyset$

Now we consider another test case, in which the advective field $\mathbf{b}$ is tangential on a part of the interface Γ.

In this case, the ARN scheme cannot work, as one cannot recover the continuity of the solution on $\Gamma^0 := \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0\}$. Instead, the choice of γ proposed in [2] is admissible.

The problem we are going to consider is

$$-\varepsilon\Delta u + bu_x = 0 \qquad \text{in } \Omega = (0,1)^2 \ ,$$

with

$$b = \begin{cases} -1 & 0 \le y \le 0.5 \\ 0 & 0.5 < y \le 1 \end{cases}$$

and boundary conditions $u = 1$ on the sides with vertex in $(0,0)$ and $u = 1$ on the sides with vertex in $(1,1)$.

In Figure 6.10 we show the number of iterations needed by the 0-DR and the ADN methods to achieve convergence, for different values of ε. We have split Ω into two parts of the same size, using a mesh with 21×21 points in each subdomain.

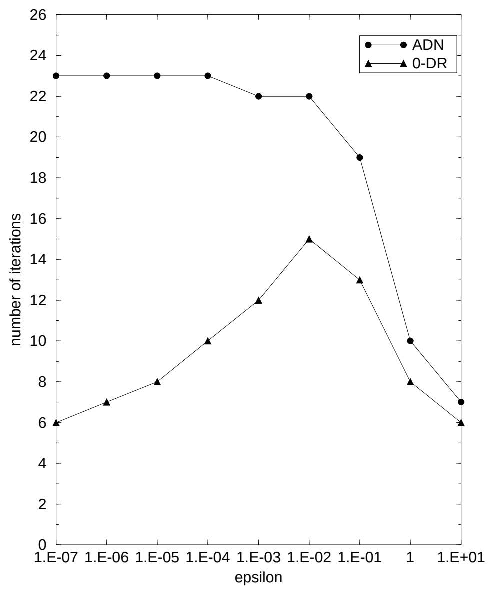

Figure 6.10. Number of iterations for ADN and 0-DR methods.

In this case the performance of the γ-RR scheme are not satisfactory, even when ε is very small. In fact, as it can be noticed in Fig. 6.11, using the value of γ given formula (4.17) the number of iterations is very large. The situation improves for the value $\gamma = \frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4\kappa\varepsilon}$ proposed by [2], for κ ranging between 10^{-2} and 10^2, but is still worse of both the 0-DR and the ADN schemes. In fact, for the case described in Fig. 6.10 and $\varepsilon = 10^{-6}$, convergence is reached after 83 iterations for $\kappa = 10^{-2}$, while 0-DR and ADN need 7 and 23 iterations, respectively. The choice of larger values of κ gives worse results.

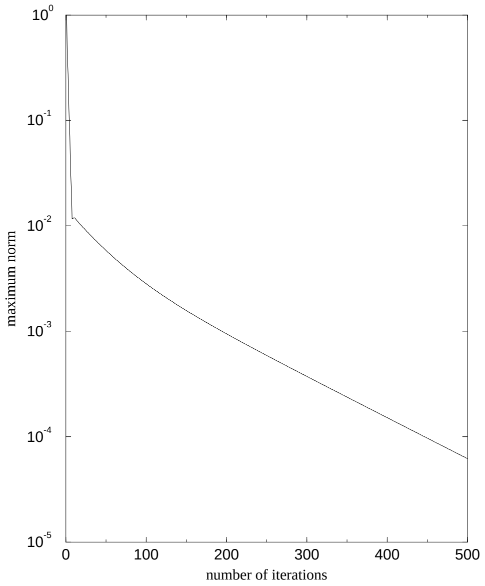

Figure 6.11. Convergence history of the γ-RR method for $\varepsilon = 10^{-5}$.

Also in this case, the number of iterations needed by the 0-DR scheme turns out to be independent of the number of nodes and ε, as is shown in Table 6.12. For these computations, we have chosen $\Omega_1 = (0, 0.25)\times(0, 1)$ and $\Omega_2 = (0.25, 1)\times(0, 1)$.

nodes \ ε	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$(11 \times 21) + (31 \times 21)$	19	19	12	15	14	15	10	11	10
$(16 \times 21) + (46 \times 21)$	19	19	12	15	14	15	10	10	10
$(21 \times 21) + (61 \times 21)$	19	19	12	16	14	15	10	10	10
$(26 \times 21) + (76 \times 21)$	19	19	12	16	14	15	10	10	9

Table 6.12. Number of iterations of the 0-DR method.

We have also applied the 0-DR and the ADN methods to other test cases, in which the advective field $\mathbf{b}$ changes direction on Γ. The performances of the 0-DR have been comparable to the ones of this third test case. On the contrary, in this situation the ADN scheme imposes a mixed Dirichlet-Neumann boundary condition on both sides of Γ, and sometimes this seems to slow down the rate of convergence, as the singularity appearing in the point where the boundary condition changes type can be propagated inside the subdomains Ω_1 and Ω_2.

7. Conclusions

We have proposed two families of domain decomposition methods for advection-diffusion equations and systems, called γ-DR and γ-RR.

Under suitable assumptions, we have proven their convergence, for both the infinite dimensional and finite dimensional cases. In particular, in the latter case the γ-DR scheme is shown to converge at a rate which is independent of the number of degrees of freedom, hence the domain decomposition procedure implicitly defines an optimal preconditioner.

We have employed these methods for computing the solution of some test problems, with good performances. The 0-DR method (namely, γ-DR for $\gamma = 0$) turns out to be particularly well-suited, as:

- it is *efficient*, as the relative error between two subsequent iterates becomes less than 10^{-10} in a few iteration-by-subdomain sweeps;
- it is *robust*, namely it can be used for large or small diffusion, with coarse or fine meshes, and in each case the rate of convergence is essentially the same. Moreover, also the relaxation parameter θ is rather insensitive to these coefficients, and the choice $\theta = 0.5$ is the optimal one provided that a uniform mesh has been used in Ω. For meshes with a different mesh-parameter in Ω_1 and Ω_2, in our computations the optimal parameter always ranges between 0.4 and 0.6, and in any case the choice $\theta = 0.5$ yields a number of iterations not far from the best one;
- it is *simple* to implement, as it doesn't require to take into account the direction of the advective field on the interface Γ for deciding the boundary condition to impose in that point (this can be rather cumbersome for non-uniform meshes). The Dirichlet boundary condition can always be used on one side of Γ, the Robin condition on the other side;
- it is *general*, namely the same algorithm can be employed also for systems of advection-diffusion equations.

Added in proof. While completing this paper, we have been aware that the γ-RR method, for any positive function γ, has been already proposed in [3]. There the authors have also proven the convergence of the subdomain iterates u_i^k in $H^1(\Omega_i)$, but only in the infinite dimensional case.

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