Coercive multidomain algorithms for incompressible fluid dynamics

Algoritmi multidominio di tipo coercivo in fluidodinamica incomprimibile

R.Loredana Trotta
Centro di Ricerca, Sviluppo e Studi Superiori in Sardegna (CRS4)
Via Nazario Sauro 10, 09123 Cagliari, Italy
Loredana.Trotta@crs4.it

1 Introduction

The aim of this work is to present a new multidomain algorithm for the solution of Navier-Stokes equations. This is obtained by applying suitable domain decomposition methods, introduced for advection dominated advection-diffusion equations, in the context of fluid dynamics equations.

The main novelty of out method [1] resides in the fact that we don't care about the local direction of the advective field on the interface Γ (as proposed in [2]), but we only need that the boundary value problems in each subdomain along the subdomain iterations are associated to a suitable coercive bilinear form.

The extention of these methods to the resolution of the Navier-Stokes equations, has been obtained by facing the numerical solution focusing on the advection-diffusion phenomena, trying to circumvent the difficulties arising form the incompressibility constraint. This can be done using suitable fractional step methods, where the pressure does not play anymore the role of a Lagrange multiplier associated to the constraint. The method used is an incremental projection method, based on a reduced number of equations with respect to the classical projection algorithms, and, in particular, can be considered as a stabilization method. This choice allows the use of equal-order interpolation spaces, and therefore yields a considerable computational saving.

The algorithms obtained are suited for parallel implementation.

2 Coercive Domain Decomposition Algorithms

Domain decomposition methods for the approximation of partial differential equations are well suited to design algorithms for parallel processing. They are based on the partition of the computational domain Ω into subregions of smaller size. The original boundary value problem is reformulated in a split form on the subdomains, and the local solutions have to satisfy suitable matching condition at subdomain interfaces. These transmission conditions are then used to set up iterative procedures among subdomains. Let us consider, for simplicity, a partition into two nonoverlapping subdomains Ω_1 and Ω_2 with a common interface $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$.

DD algorithms that work well for viscosity flows can perform very badly for convectively dominated flows; these algorithms may produce, at each iteration, nonphysical layers at the outflow part of the interface. This is due to the fact that these methods do not take care of the direction of the flow.

Methods that take into account the local flow direction in a way to have an optimal range for the choice of the relaxation parameter θ, and that prevent the rise of artificial layers at subdomain interfaces as the advection becomes dominant, have been considered in the past [2].

The new family of methods, that we have developed, is easier to implement, as it doesn't require to take into account the direction of the advective field on the interface Γ to decide the boundary condition in that point (this can be rather cumbersome for non-uniform meshes). The choice that leads to efficient iteration-by-subdomain schemes is the one which, in each subdomain, preserves the *coerciveness* of the associated bilinear form.

One of these methods, the **0-DR** iterative scheme, starting from λ^0, construct the iterates u_i^k $(i = 1, 2)$ by solving the following advection-diffusion problems:

$$\begin{cases} -\varepsilon \Delta u_i^{k+1} + \mathrm{div}(\mathbf{b} u_i^{k+1}) + a u_i^{k+1} = f & \text{in each } \Omega_i \\ \begin{cases} u_1^{k+1} = \lambda^k & \text{pbl in } \Omega_1 \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial \mathbf{n}} - \dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} u_2^{k+1} = \varepsilon \dfrac{\partial u_1^{k}}{\partial \mathbf{n}} - \dfrac{1}{2}\mathbf{b}\cdot\mathbf{n} u_1^{k} & \text{pbl in } \Omega_2 \end{cases} & \text{on } \Gamma \end{cases}$$

where $\lambda^{k+1} = \theta u_2^{k+1} + (1-\theta)\lambda^k$ on Γ.

3 Projection methods

The motion of an incompressible viscous fluid is described by the Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \Big(2\nu \mathbf{D}(\mathbf{u})\Big) + \mathcal{B}(\mathbf{u};\mathbf{u}) = -\nabla p + \mathbf{f} & \text{in } \Omega \times (0,T) \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega \times (0,T)\,, \end{cases} \tag{1}$$

where

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\,, \qquad \mathcal{B}(\mathbf{u};\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\mathrm{div}\,\mathbf{u})\mathbf{v}\,,$$

$\mathbf{u}$ and p are the unknown velocity field and pressure, respectively, ν is the viscosity and $\mathbf{f}$ the applied body force.

The incompressibility constraint couples velocity and pressure, which is a variable that adjusts itself instantaneously in a way that the incompressibility constraint is always verified. The major computational difficulties in devising an efficient code are related to the incompressibility constraint and to the convective-dominant character of the non-linear momentum equations.

The projection methods are particular fractional step methods, based on the decomposition theorem of Hodge-Ladyzhenskaya, and originally proposed by Chorin and Temam. Their goal is to allow the separate treatment of the difficulties related to the nonlinear convection and to the incompressibility constraint. The solution is obtained as the limit of a sequence of simpler advection-diffusion problems for an "intermediate" velocity $\tilde{\mathbf{u}}$, not satisfying the condition of incompressibility. This solution $\tilde{\mathbf{u}}$ verifies the same boundary conditions of the original problem, and the gradient of the pressure is either omitted (*non incremental method*), or explicitly treated (*incremental method*). The intermediate field $\tilde{\mathbf{u}}$ is then decomposed into the sum of a solenoidal velocity field, and the gradient of a scalar function proportional to the pressure.

Using, instead of the projection step, its equivalent form, given as a Poisson equation for the pressure, we have noticed that the benefit of the projection method is therefore more than just uncoupling the velocity and the pressure computations. In this way the projection method can be re-interpreted as a suitable pressure stabilization method of pseudo-compressibility type, for which a well developed convergence analysis is available. Writing the whole algorithm in term of the intermediate velocity $\tilde{\mathbf{u}}$, we have found two different consequences: the simplicity of the scheme, and the fact that in this way the theoretical stabilization properties intrinsic to the projection methods become numerically effective.

For the problem (1) subjected to the Dirichlet boundary condition $\mathbf{u} = \mathbf{w}$, the method we have used [3] corresponds to solve the following problems:

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^{n}}{\Delta t} - \nabla\Big(2\nu\mathbf{D}(\tilde{\mathbf{u}}^{n+1})\Big) + \mathcal{B}(\tilde{\mathbf{u}}^{n};\tilde{\mathbf{u}}^{n+1}) = \mathbf{f}^{n+1} - \nabla\Big((\gamma+1)p^{n} - \gamma p^{n-1}\Big) \\ \tilde{\mathbf{u}}^{n+1}_{|\partial\Omega} = \mathbf{w}^{n+1}_{|\partial\Omega}\,, \end{cases}$$

$$\begin{cases} \Delta\varphi^{n+1} = \dfrac{1}{\Delta t}\nabla\cdot\tilde{\mathbf{u}}^{n+1} \\ \dfrac{\partial\varphi^{n+1}}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0\,, \end{cases} \tag{2}$$

where for $\gamma = 0$ we have the *non incremetal* scheme, while $\gamma = 1$ characterizes the *incremental* one.

4 Validation on test cases

Knowing the domain decomposition formulation for convective-diffusion problems, we have also considered the solution of the Navier-Stokes by means of domain decomposition algorithms. In particular, applying the projection method here presented, we can speed up the convergence using the coercive methods for the vectorial advection diffusion equations (2). Furthermore the pressure solver can incorporate a standard “elliptic” method, such as the Dirichlet/Neumann one.

The experiments made to test the capability of the coercive algorithms have been done using a finite element discretization, and the schemes are implemented on a cluster of IBM RS/6000 workstations connected by Ethernet.

In Figure 1 we plot the velocity solution on the sections in the middle of the unitary cavity and the streamlines in the case $Re = 3200$. The solution coincides with those obtained solving the problem on the whole computational domain Ω. The projection method reproduces the “exact” Ghia solution, and the coercive domain decomposition method does not introduce sensitivity to the mesh size h.

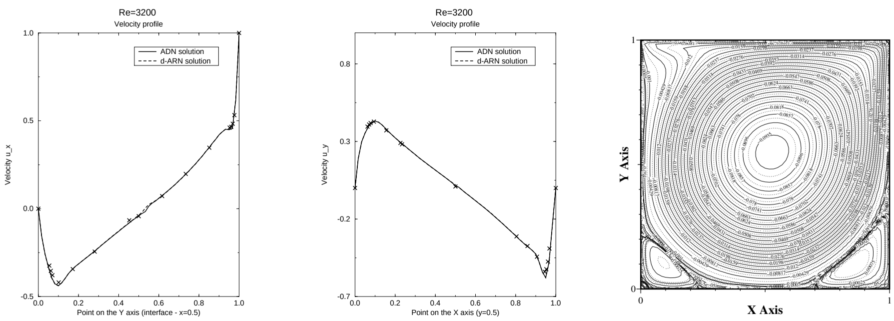

Figure 1: Solution obtained for the cavity at $Re = 3200$ and streamlines

REFERENCES

1. A.Alonso, R.L.Trotta and A.Valli, Coercive Domain Decomposition Algorithms for Advection-Diffusion Equations and Systems, UTM 526, January 1998, Trento. Submitted to “Journal of Computational and Applied Mathematics”, North-Holland, Amsterdam

2. F. Gastaldi, L. Gastaldi and A. Quarteroni, Adaptive domain decomposition methods for advection dominated equations, East-West J. Numer. Math. **3**, 165–206 (1996)

3. R.L.Trotta, Metodi di proiezione, di stabilizzazione e di decomposizione di dominio in fluidodinamica incomprimibile laminare e turbolenta, Tesi di Dottorato, Trento, 1997.