Parallel spectral elements for applications to linear elastodynamics

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Abstract

We present a parallel implementation of a fully unstructured spectral element method for elastic wave propagation problems. Our target is to merge the high accuracy of spectral methods with the flexibility of finite elements in dealing with complex geometries. The spectral algorithm we propose is based on a domain decomposition approach, implemented into a multiprocessor environment. Parallelism is exploited at the algebraic level: the global linear system is distributed among processors, and then solved by iterative techniques.

1 Introduction

In recent years, the spectral element method has been applied to the numerical solution of elastic wave propagation problems, thus enabling significant reductions of computational resources respect to more traditional techniques like finite elements or finite differences (see for instance [13], [4], [5], and [9]). The main advantage of spectral elements is their high order approach, which equals the accuracy of low order methods using less number of grid points per wavelength. This is an appealing feature in real case problems, often unaffordable because of the exceedingly large amount of computational effort required. On the other hand, spectral elements are not so flexible as finite elements in dealing with complex geometries. More in details, finite element approach to wave simulation can benefit from a well established technology consisting in a variety of tools, like efficient mesh generators and fast algebraic solvers, developed in the course of the last two decades for the treatment of advanced applications in structural engineering and fluid-dynamics. The object of this work is to implement an efficient parallel spectral element method for the elastodynamic equation. The algorithm should enjoy the flexibility of finite element solvers, producing a framework capable of dealing with CAD-oriented geometries, and interaction with the best-known engineering tools. On the other hand, we resort to coarse parallelism adopting a fully unstructured multiprocessor paradigm. This approach is based on the balanced decomposition of the algebraic problem among subdomains.

This is the natural development of previous work aimed at the theoretical analysis of the spectral approach for wave propagation problems (see [4], [5]); in [10], [1], [2], and [3] a more general approach, based on the *mortar* coupling between finite and spectral elements and aimed to soil-structure interaction and non-linear problems, has already been investigated and implemented in sequential form.

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2 Problem formulation

The equation of motion, in variational form, for an elastic bounded medium $\Omega \subset \mathbb{R}^d$, ($d = 2, 3$ is the number of space dimensions), subject to an external force distribution $\mathbf{f}^{\text{ext}}$ reads: $\forall t > 0$, find $\mathbf{u}(t) \in V$ such that $\mathbf{u}(t) = \mathbf{g}(t)$ on Γ_D, and for any $\mathbf{v} \in V$,

$$\int_\Omega \rho \frac{\partial^2}{\partial t^2}\mathbf{u} \cdot \mathbf{v} \, d\Omega + \int_\Omega \underline{\sigma}(\mathbf{u}) : \underline{\epsilon}(\mathbf{v}) \, d\Omega =$$

$$\int_\Omega \mathbf{f}^{\text{ext}} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v} \, d\Gamma + \int_{\Gamma_{NR}} \mathbf{t}^* \cdot \mathbf{v} \, d\Gamma \ , \tag{1}$$

where V is a suitable space of admissible displacements (see [5]), $\underline{\sigma}$ and $\underline{\epsilon}$ are the stress and strain tensors, related through the Hooke's law. Γ_D and Γ_N are those parts of the boundary where displacement and tractions are prescribed, respectively, while on Γ_{NR} non-reflecting conditions are set. The latter are designed for simulating propagation in unbounded domains (see [15]). Initial conditions on $\mathbf{u}$ and $\dot{\mathbf{u}}$ should also be provided. Since our approach is based on the solution of the global algebraic problem, it easily allows to address the static case, for which the first term in the left hand side of (1) vanishes (and there is no need of non-reflecting conditions).
The equation of motion (1) is then discretized by means of a suitable finite dimensional space (see [5] and [3]), thus obtaining a set of second order ordinary differential equations:

$$\underline{M}\ddot{\mathbf{U}} + \underline{C}\dot{\mathbf{U}} + \underline{K}\mathbf{U} = \mathbf{F}^{\text{ext}} \ , \tag{2}$$

where $\underline{M}$, $\underline{C}$ and $\underline{K}$ are the *mass*, *damping* and *stiffness* matrix, respectively (the damping matrix arises from non-reflecting conditions or, eventually, from the presence of viscous media), $\mathbf{U}$ is the vector of unknown displacements, along with its time-derivatives, and $\mathbf{F}^{\text{ext}}$ is the vector of externally applied loads.

3 Space discretization and grid generation

Spectral elements are a generalization of parallelepipedal finite elements, exploiting space discretization by quadrilaterals (2D) or hexahedra (3D), and based on the use of high order polynomial spaces (see [14]). The degree of polynomials used for approximating the solution of (1), which can be fixed at run-time by the user, and the linear size of the geometrical grid, determine the accuracy of the numerical solution, and should be properly chosen. In particular, setting polynomial degree to 1, makes our method undistinguished from finite elements.
Following the standard procedure, a geometry-model is first created and then discretized by means of a grid of quadrilaterals or hexahedra. While the construction of pure quadrileral meshes is easily performed by the most common grid generators, the 3D case is more delicate. Grids of hexahedra can be generated as mapped meshes, or via decomposition of tetrahedra or direct methods (see [12] for an interesting review on the argument). Once the physical domain has been discretized by the grid, and the polynomial degree has been chosen, the spectral nodes are built, using a suitable mapping with a parent geometry (a square or a cube); this process is carried on element-by-element, as depicted in Figure 1. The computation of spectral nodes is done by the solver rather than by the mesh generator; thus, any qudrilateral or hexahedra grid in finite element format can be used as an input, and the dimensions of the files containing the mesh do not exceed reasonable limits.

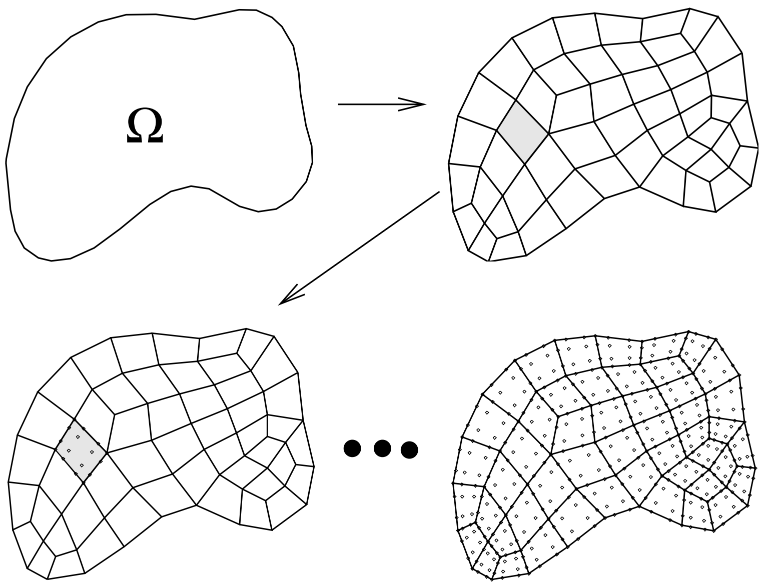

FIG. 1. *Mesh generation and definition of spectral nodes*

4 Domain decomposition and parallel spectral elements

The discretization of first and second order time-derivatives in (2) by means of backward finite-difference schemes (see [11]), gives rise to the linear algebraic system

$$\underline{A}\mathbf{U}^{k+1} = \mathbf{b}^{k+1} , \tag{3}$$

where $\underline{A}$ is a combination of the mass, damping, and stiffness matrices, $\mathbf{U}^{k+1}$ are the unknown displacements at time $t_{k+1} = (k+1)\Delta t$ (Δt is the time step), and the right hand side $\mathbf{b}^{k+1}$ can be evaluated using the previous displacements and the known external forces. Problem (3) can be solved (at each time step) using parallel iterative procedures. Our approach stems from the decomposition of the physical domain Ω in a number of subdomains: this task is not trivial since the produced partition should be well balanced both in terms of number of degrees of freedom, and in terms of communication to be performed among subdomains during the parallel solution of (3). The partition operation is done by Metis (see [8]), an advanced software package allowing the efficient domain decomposition of large unstructured meshes. Once the domain Ω has been split, each subdomain is managed by a different processor of a parallel machine, which should be able to build that portion of the spectral matrix $\underline{A}$ of (3) corresponding to the nodes belonging to the subdomain in object. In order to do that, information is exchanged among processors via an MPI procedure ([6]). At this stage the parallel solution of the distributed algebraic problem is demanded to Aztec (see [7]), a fast library for the parallel iterative solution of large linear systems, which also takes care of the communication needed by the algebraic solver, and provide the global solution. This process is shown in Figure 2.

5 Numerical example

We show a simple example concerning elastic wave propagation through a 2D domain with circular profile and an irregular internal hole. The circumference has a radius 1000 m long and is filled with homogeneous material corresponding to a density $\rho = 3200$ kg·m^{-3}, and to pressure and shear velocities given, respectively, by $\alpha = 1750$ m·s^{-1} and $\beta = 1200$ m·s^{-1}.

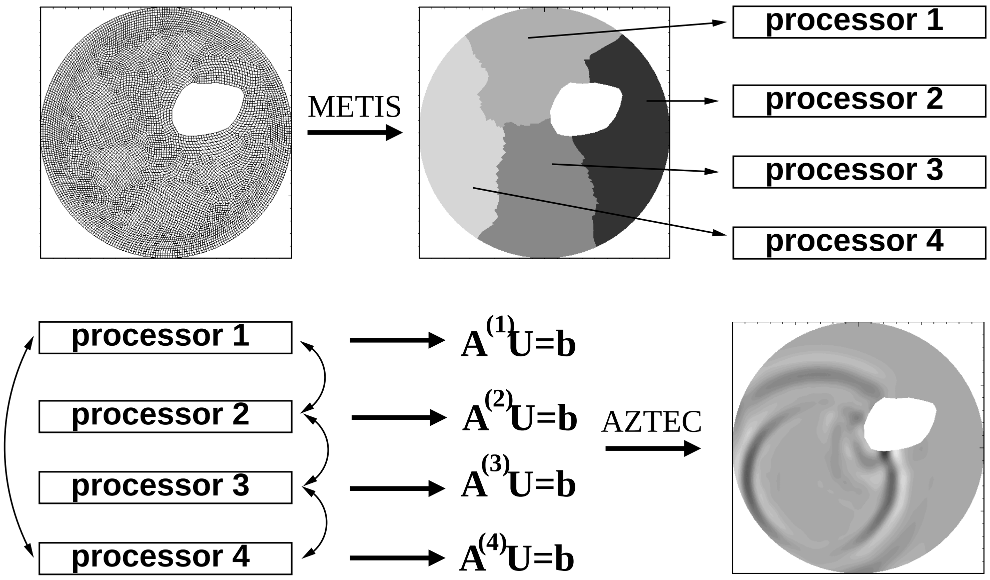

FIG. 2. *Top: Domain decomposition and distribution on different processors. Bottom: generation of distributed matrices and parallel solution of the algebraic system*

A point source corresponding to a vertical body-force has been placed at the position with coordinates -311.3 m, -246.4 m with respect to the center of the domain; its time-history is a Richer wavelet with a peak frequency of 17 Hz. Non-reflecting conditions have been set to the external boundary, while the profile of the internal hole is treated as a free-surface ($\mathbf{t} = \mathbf{0}$ in (1)). Figure 3 shows three snapshots of the wave propagation, corresponding to $t = 0.5$ s, $t = 0.6$ s and $t = 0.7$ s, respectively.

References

[1] Casadei, F., and E. Gabellini, *Implementation of a Coupled Spectral Element/Finite Element Solver for Wave Propagation and Soil-Structure Interaction Simulations. Part I - Models*, EUR 17730 EN, 1997

[2] Casadei, F., and E. Gabellini, *Implementation of a Coupled Spectral Element/Finite Element Solver for Wave Propagation and Soil-Structure Interaction Simulations. Part II - Numerical Examples*, EUR 18051 EN, 1998

[3] Casadei, F., E. Gabellini, G. Fotia, F. Maggio, and A. Quarteroni, *A hybrid spectral-element/finite-element method for wave propagation and soil-structure interaction* (in preparation)

[4] Faccioli, E., F. Maggio, A. Quarteroni, and A. Tagliani, *Spectral-domain decomposition methods for the solution of acoustic and elastic wave equations*, Geophysics, 61 (4), pp. 1160-1174, 1996

[5] Faccioli, E., F. Maggio, R. Paolucci, and A. Quarteroni, *2D and elastic wave propagation by a pseudo-spectral domain decomposition method*, Journal of Seismology, 1, 237–251, 1997

[6] Gropp, W., E. Lusk and A. Skjellum, *Using MPI*, The MIT Press, 1994

[7] Hutchinson, S.A., L.V. Prevost, J,N. Shadid, C. Tong, and R.S. Tuminaro, *Aztec User's Guide - Version 2.0* (http://www.cs.sandia.gov/CRF/aztec1.html), 1998

[8] Karypis, G., and V. Kumar, *METIS - A Software Package for Partitioning Unstructured Graphs, Partitioning Meshes, and Computing Fill-Reducing Orderings of Sparse Matrices*, (User's manual, http://www-users.cs.umn.edu/ karypis/metis/), 1998

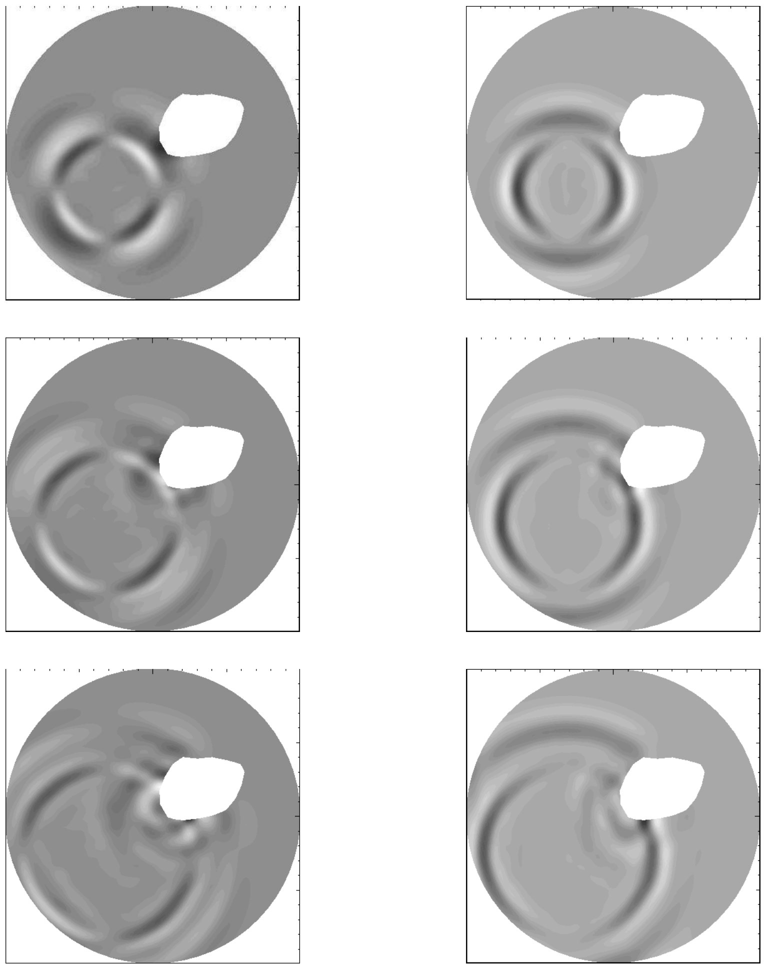

FIG. 3. *Wave propagation through a 2D irregular geometry. Left column: horizontal displacement; right column: vertical displacement*

[9] Komatitsch, D., and J.P. Vilotte, *The spectral element method: an efficient tool to simulate the seismic response of 2D and 3D geological structures*, Bulletin of the Seismological Society of America, 88, pp. 368-392, 1998

[10] Lahaye, D.J.P., F. Maggio, and A. Quarteroni, *Hybrid finite element-spectral element approximation of wave propagation problems*, East-West Journal of Numerical Mathematics, Vol.5, No.4, 1997

[11] Maggio, F., and A. Quarteroni, *Acoustic wave simulation by spectral methods*, East-West Journal of Numerical Mathematics, 2 (2), pp. 129-150, 1994

[12] Owen, S.J., *A Survey on Unstructured Mesh Generation Technology*, Proceedings 7*th* International Meshing Roundtable, Dearborn, MI (http://www.andrew.cmu.edu/user/sowen/)

[13] Priolo, E., and G. Seriani, *A numerical investigation of Chebychev spectral element method for acoustic waves propagation*, in Vichnevetsky, R., ed. Proc. 13th IMACS World Congress on Computational and Applied Mathematics, Criterion Press, pp. 551-556, 1991

[14] Quarteroni, A., and A. Valli, *Numerical approximation of partial differential equations*, Springer Verlag, Berlin, 1994

[15] Stacey, R., *Improved Transparent Boundary Formulations for the Elastic-Wave Equation*, Short Note, Bulletin of the Seismological Society of America, 78 (6), pp. 2089-2097, 1988