Spectral methods in non stationary thermoelasticity: first results

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1 Introduction

Our interest in the applications of thermoelasticity is mainly due to the research activities in the field of beam dumping facility design for particle accelerators, in particular for the new LHC at CERN.
Interaction between high energy particle beams with targets is responsible for sudden and concentrated heating of the material of the targets, this give rise to stress waves propagating inside their structure, that in some cases may be dangerous for structure stability. Well focused beams with high energy produce more intense stress waves with a smaller characteristic wavelength, the simulation of these phenomena becomes more important but also more challenging for the computational effort involved. We have therefore applied the spectral element technology developed at CRS4 to this particular class of problems, the resulting tools have already been used, in an early stage, for a case study for the TOF experiment at CERN [1].
Anyway, deformations and stresses due to temperature gradients are common in almost all engineering applications, and thermomechanical analyses are common practice in design. While most of the normal applications may be solved under the hypothesis of stationarity for the elastic or the thermal part of the problem, there is a class of applications for which this kind of approximation is not admissible and a transient solution of the thermal and elastic problems is required. Apart from the afore mentioned beam dumping problem, examples may be found in almost every branch of engineering, and their study has long been a subject matter of widespread research activity and interest, mainly in the field of classical thermoelastic effects and recently also in the field of non classical heat transmission models and thermomechanical interaction. A review of recent research activities may be found in [2] and [3].
In this report a basic formulation of the general transient thermoelastic problem will be given, then the solution method, using spectral elements, will be described, and finally a simple reference test case will be analyzed.

2 Problem formulation

We consider first a general formulation of a coupled thermoelastic problem.
Consider a body occupying at time (t = 0) the finite region $\Omega \subset R^d$ (with d = 2 or 3, number of space dimensions) with boundary Γ.
The body shall satisfy the momentum balance equation and the energy balance equation:

$$\rho \ddot{u}_i = \sigma_{ji,j} + \rho f_i \qquad \text{in } \Omega \times (0, T_f) \qquad \forall i = 1, \ldots, d \tag{1}$$

$$\rho c \dot{\theta} = \rho Q - q_{i,i} - T_0 \gamma \dot{e} \qquad \text{in } \Omega \times (0, T_f) \tag{2}$$

The body status is described by the values of the displacements u_i and the relative temperature $\theta = T - T_0$ being T_0 the stress free initial temperature expressed in the Kelvin scale.

The usual notation is used for the vector components and their derivatives, time derivatives are denoted by overlying dots, and repeated indices mean summation.
In the momentum balance equation σ_{ij} is the tensor of the Cauchy stresses, f_i are the components of external distributed body forces and ρ is the density. In the energy balance equation c is the specific heat, Q is the external heat source per unit area, or volume, q_i are the components of the heat flux.
The body is supposed to be a linear isotropic elastic solid, and to conduct heat according to the Fourier's law:

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu \epsilon_{ij} - \gamma \theta \delta_{ij} \tag{3}$$

$$q_i = -k\theta_{,i} \tag{4}$$

The first is the Hooke's law expressed by means of the Lamé coefficients λ and μ, ϵ_{ij} is the deformations tensor. In the second equation k is the heat conductivity. We have

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

$$e = div(\mathbf{u}) \tag{6}$$

$$\gamma = \alpha(3\lambda + 2\mu) \tag{7}$$

The coupling between the standard elastodynamics problem and the thermal problem is given by the last term in the elastic constitutive law, representing the stresses due to the thermal expansions, and by the last term in the energy balance equation accounting for the internal heat source due to the rate of elastic volumetric deformation.
The boundary Γ is subjected to mechanical and thermal boundary conditions:

$$\Gamma = \Gamma_D^{el} \cup \Gamma_N^{el} \cup \Gamma_A^{el} = \Gamma_D^{th} \cup \Gamma_N^{th} \cup \Gamma_C^{th} \tag{8}$$

with:

$$\Gamma_D^{el} \cap \Gamma_N^{el} \cap \Gamma_A^{el} = 0 \tag{9}$$

$$\Gamma_D^{th} \cap \Gamma_N^{th} \cap \Gamma_C^{th} = 0 \tag{10}$$

With reference to the thermal problem, the computational domain boundary Γ is divided in three non overlapping parts: Γ_D^{th} where Dirichlet boundary conditions are applied, Γ_N^{th} where the boundary is of the Neumann type and Γ_C^{th} that represents and heat convection surface, where heat flux depends from an external medium temperature, an heat exchange coefficient, but also from the surface temperature value. No thermal radiation is included since this would imply a non linear condition. In the following only the Dirichlet and Neumann boundary conditions will be discussed, for the sake of simplicity:

$$\theta = \theta_D \qquad \text{on } \Gamma_D^{th} \times (0, T_f) \tag{11}$$

$$q_i n_i = q_N \qquad \text{on } \Gamma_N^{th} \times (0, T_f) \tag{12}$$

It is also necessary to specify the initial conditions for the temperature in Ω:

$$\theta(t = 0) = \theta^{(0)} \qquad \text{in } \Omega \tag{13}$$

The boundary is also subjected to elastic boundary conditions, expressed in the form:

$$u_i = \phi_i \qquad \text{on } \Gamma_{Di}^{el} \times (0, T_f) \qquad i = 1, \ldots, d \tag{14}$$

$$\sigma_{ij} n_j = t_i \qquad \text{on } \Gamma_{Ni}^{el} \times (0, T_f) \qquad i = 1, \ldots, d \tag{15}$$

On Γ_{Di}^{el} prescribed values of the i component of displacements are imposed, on Γ_{Ni}^{el} known tractions are applied. Apart from these an absorbing boundary condition has been implemented, used to approximate the infinite media behavior; this will not be discussed here and the reader may refer to [4][5] for details on this subject. The initial conditions for the elastodynamics problem should be provided as well:

$$\begin{aligned} u_i(t=0) &= u_i^{(0)} \quad \text{in } \Omega && (16)\\ \dot{u}_i(t=0) &= \dot{u}_i^{(0)} \quad \text{in } \Omega && (17) \end{aligned}$$

Through the principle of virtual work, the dynamic equilibrium problem can be stated in the following *weak* or variational form:
if $\bar{\mathbf{u}}$ is generic function candidate to represent a possible solution and vanishing on Γ_D^{el}, find $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ such that $\forall t \in (0, T_f)$ and $\forall \bar{\mathbf{u}}$:

$$\int_\Omega \rho \bar{u}_i \ddot{u}_i \, d\Omega + \int_\Omega \bar{u}_{i,j} \sigma_{ij} \, d\Omega = \int_\Omega \bar{u}_i f_i \, d\Omega + \int_{\Gamma_N^{el}} \bar{u}_i t_i \, d\Gamma \tag{18}$$

Similarly for the thermal problem we denote by $\bar{\theta}$ a generic function candidate to represent a possible solution and vanishing on Γ_D^{th}, and we look for $\theta = \theta(\mathbf{x}, t)$ such that $\forall t \in (0, T_f)$ and $\forall \bar{\theta}$:

$$\int_\Omega \bar{\theta} \rho c \dot{\theta} \, d\Omega + \int_\Omega \bar{\theta}_{,i} k \theta_{,i} \, d\Omega = \int_\Omega \bar{\theta} \rho Q \, d\Omega - \int_\Omega \bar{\theta} T_0 \gamma \dot{e} \, d\Omega + \int_{\Gamma_N^{th}} \bar{\theta} q_N \, d\Gamma \tag{19}$$

if we split the thermal component of the stress:

$$\begin{aligned} \sigma_{ij} &= \sigma_{ij}^* - \sigma_{ij}^{th} && (20)\\ \sigma_{ij}^* &= \lambda e \delta_{ij} + 2\mu \epsilon_{ij} && (21)\\ \sigma_{ij}^{th} &= \gamma \theta \delta_{ij} && (22) \end{aligned}$$

then the we can formulate the thermoelastic problem as follows:
Find $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\theta = \theta(\mathbf{x}, t)$ such that $\forall t \in (0, T_f)$ and $\forall \bar{\mathbf{u}}$, $\forall \bar{\theta}$:

$$\begin{aligned} \int_\Omega \rho \bar{u}_i \ddot{u}_i \, d\Omega + \int_\Omega \bar{u}_{i,j} \sigma_{ij}^* \, d\Omega &= \int_\Omega \bar{u}_i f_i \, d\Omega + \int_\Omega \bar{u}_{i,j} \sigma_{ij}^{th} \, d\Omega + \int_{\Gamma_N^{el}} \bar{u}_i t_i \, d\Gamma && (23)\\ \int_\Omega \bar{\theta} \rho c \dot{\theta} \, d\Omega + \int_\Omega \bar{\theta}_{,i} k_i \theta_{,i} \, d\Omega &= \int_\Omega \bar{\theta} \rho Q \, d\Omega - \int_\Omega \bar{\theta} T_0 \gamma \dot{e} \, d\Omega + \int_{\Gamma_N^{th}} \bar{\theta} q^\Gamma \, d\Gamma && (24) \end{aligned}$$

This problem has been solved numerically using different techniques, including the well known finite differences and finite element method. In the following the spectral element method will be briefly discussed and successfully applied.

3 Spectral approximation

The Spectral Element Method (SEM) may be considered as a generalization of standard finite element method, using high order piecewise polynomial functions [4][5].
It is an *h-p* method, since the accuracy of the numerical solution may be increased through a mesh refinement (reducing element mean size) as in a standard finite element method, or enhancing the algebraic degree of the functions. This latter opportunity results very useful to the final user who simply has to set the desired spectral degree at runtime to obtain the needed accuracy, whereas no time consuming model meshing or re-meshing process is required.
Being u the "exact" regular solution, then the numerical solution $u_{H,n}$ will converge to the exact value according to the following law:

$$\| u - u_{H,n} \| \leq C H^n exp(-n) \tag{25}$$

where $\| \cdot \|$ is th H^1 norm and the constant C is independent from element size H and spectral degree n.

To obtain an approximate numerical solution for the thermal and elastic problems we divide the computational domain Ω in a number of subdomains, the spectral elements, that are quadrilaterals in 2D problems and hexaedra in 3D, denoted by $\Omega_1, \Omega_2, \dots, \Omega_K$.
Each element Ω_k is obtained mapping a reference (or "master") domain $\widehat{\Omega} = [-1,1]^d$, $(d = 2,3)$ with a transformation $\mathbf{x} = \mathbf{F}_k(\hat{\mathbf{x}})$ where $\mathbf{x}$ and $\hat{\mathbf{x}}$ refer to Ω_k and $\widehat{\Omega}$, respectively.
The method is not influenced by the nature of the transformation adopted, as long as this is a 1 to 1 mapping with a constant in sign and non zero Jacobian determinant, and the mapping provides the same node positioning on the sub-domain interfaces.
The simplest transformation, adopted in the code, is the Q_1 mapping that associates the vertex of Ω_k with those of $\widehat{\Omega}$ and is linear in each space coordinate. The transformation is sub-parametric, but this does not affect the convergence behavior, and positively reduces the grid generation effort.
The space of admissible displacement components and temperatures ($\bar{u}_i$ and $\bar{\theta}$ in equations (18) and (19)) is defined as $\mathbf{V}^0 = \left\{ v \in \mathbf{C}^0(\Omega) \,|\, v|_{\Omega_k} \in \mathbf{Q}_N(\Omega_k) \, and \, v|_{\Gamma_D} = 0 \right\}$ where $\mathbf{Q}_N(\Omega_k) = \left\{ v = \hat{v} \circ \mathbf{F}_k^{-1}, \hat{v} \in \mathbf{Q}_N(\widehat{\Omega}) \right\}$, being $\mathbf{Q}_N(\widehat{\Omega})$ the space of polynomials in the variables $(x_1, \dots, x_d)$ with degree less than or equal to N, with respect to each variable; the solution is expressed as

$$f_N(\mathbf{x}, t) = \tilde{f}_q(t) \psi_q(\mathbf{x}) \tag{26}$$

where $\tilde{f_q}(t)$ are the actual unknowns of our problem, being either a component of displacement vector, or the temperature. The ψ_q's are piecewise Lagrange polynomials of degree N which are equal to one at the q-th node and vanish at all other nodes ($\tilde{f_q}(t) = f_N(\mathbf{x}_q, t)$).
The nodes are defined in the reference domain and mapped onto each of the elements Ω_k; transformation functions are chosen so that adjacent subdomains have nodes mapped in the same positions at their interface (with no duplicate nodes), and the continuity of the solution between subdomains is easily guaranteed, since facing elements have interface nodes in common.
The nodes of the reference domain are defined as the roots of the polynomial obtained by the product of d one dimensional polynomials of the form $(1-\xi)(1+\xi)L'_N(\xi)$, where L_N is the Legendre polynomial of degree N and ξ is the coordinate in the reference domain [-1,1].
The integrals in the previous weak form of the thermal and elastic problems are be performed on each element Ω_k and the results is then summed. On the elements the integrals are evaluated numerically by a suitable approximation of high accuracy, namely the Legendre-Gauss-Lobatto (LGL) quadrature formula:

$$\int_{\Omega_k} f \, d\Omega \simeq \sum_q f(\mathbf{x}_q^{(k)}) \omega_q^{(k)} \tag{27}$$

where $\mathbf{x}_j(k)$ is the vector of coordinates of the LGL nodes in Ω_k and the ω_q are the corresponding weights.
According to the well known strategy of the standard finite element procedures, integrations are performed over the reference domain $\widehat{\Omega}$ so the previous integral becomes:

$$\int_{\Omega_k} f \, d\Omega \simeq \sum_q f(\hat{\mathbf{x}}_q^{(k)}) J_k(\hat{\mathbf{x}}_q) \hat{\omega}_q \tag{28}$$

where $\mathbf{x}_q$ are the images of $\hat{\mathbf{x}}_q$ through $\mathbf{F}_k$ and J_k is the determinant of the Jacobian of $\mathbf{F}_k$.
This choice preserves the exponential accuracy, with the error of the numerical solution decaying exponentially with increasing N (see (25), and the computational effort in evaluating the integrals and solving the resulting system is minimized since matrices for inertia and heat capacity terms are naturally diagonal.

4 Matrix formulation

In the previous equations (23) and (24) the thermal and elastic problem was expressed in the weak form, there the *virtual* or *test* displacement $\bar{\mathbf{u}}$ and temperature $\bar{\theta}$ may be chosen arbitrarily in the functional space of the test functions. Considering a set of test functions for the displacements and temperature, each of them being equal to unity for a specific degree of freedom of a node, the thermoelastic problem may be formulated in a matrix form:

$$\mathbf{M}^{mm}\ddot{\mathbf{u}} + \mathbf{K}^{mm}\mathbf{u} = \mathbf{f}^{mm} + \mathbf{f}^{m\theta} \tag{29}$$

$$\mathbf{C}^{\theta\theta}\dot{\theta} + \mathbf{K}^{\theta\theta}\theta = \mathbf{f}^{\theta\theta} + \mathbf{f}^{\theta m} \tag{30}$$

where $\mathbf{u}$ is the vector of nodal displacements; $\ddot{\mathbf{u}}$ the vector of nodal accelerations; θ and $\dot{\theta}$ are the vectors of nodal temperatures and temperature variation in time.
In the matrix notation superscripts are used, m and θ denote mechancal and thermal properties, respectively. Therefore $\mathbf{M}^{mm}$ is the inertia matrix; $\mathbf{K}^{mm}$ the stiffness matrix; $\mathbf{C}^{\theta\theta}$ the heat capacity matrix and $\mathbf{K}^{\theta\theta}$ the heat conductivity matrix.
The vector $\mathbf{f}^{mm}$ includes mechanical loads distributed on the body and on the boundary, while $\mathbf{f}^{m\theta}$ is the load vector due to thermal deformation. For the thermal problem $\mathbf{f}^{\theta\theta}$ includes distributed and boundary heat loads; while $\mathbf{f}^{\theta m}$ includes the thermal energy term due to the Biot's coupling effect.
The matrices are obtained with an assembly procedure of the element matrices $\mathbf{M}$, as well as the load vectors.
The element matrices and vectors result in the following, where p and q are indices for the nodes, i, j and l are indices for vector component, and we do not make use of the repeated indices rule for summations:

$$M^{mm(k)}_{piqj} = \delta_{ij} \int_{\Omega_k} \psi_{pi}\rho\psi_{qj}\, d\Omega \tag{31}$$

$$K^{mm(k)}_{piqj} = \int_{\Omega_k} \psi_{pi,j}(\delta_{ij}\lambda\psi_{ql,l} + \mu(\psi_{qi,j} + \psi_{qj,i}))\, d\Omega \quad \text{sum on } l = 1, \ldots, d \tag{32}$$

$$f^{mm(k)}_{pi} = \int_{\Omega_k} \psi_{pi} f_i\, d\Omega + \int_{\Gamma^{el}_{N_k}} \psi_{pi} {t_i}^{\Gamma}\, d\Gamma \tag{33}$$

$$f^{m\theta(k)}_{pi} = \int_{\Omega_k} \psi_{pi}\gamma\theta\, d\Omega \tag{34}$$

$$C^{\theta\theta(k)}_{pq} = \int_{\Omega_k} \psi_p \rho c \psi_q\, d\Omega \tag{35}$$

$$K^{\theta\theta(k)}_{pq} = \int_{\Omega_k} \psi_{p,l} k \psi_{q,l}\, d\Omega \qquad \text{sum on } l = 1, \ldots, d \tag{36}$$

$$f^{\theta\theta(k)}_{p} = \int_{\Omega_k} \psi_p \rho Q\, d\Omega + \int_{\Gamma^{th}_{N_k}} \psi_p q_N\, d\Gamma \tag{37}$$

$$f^{\theta m(k)}_{p} = \int_{\Omega_k} \psi_p T_0 \gamma \dot{e}\, d\Omega \tag{38}$$

The elastic problem and the thermal problem are coupled by the $\mathbf{f}^{m\theta}$ and $\mathbf{f}^{\theta m}$ vectors. The values of all the vectors $\mathbf{f}$ have to be calculated at each time step while the matrices remain constant due to the linearity of the problem.
In addition to space discretisation, it is necessary to specify a time advancing algorithm to allow the solution of the coupled system.

5 Time discretisation

Several approaches have been proposed in literature to solve the coupled thermoelastic problem with implicit and explicit schemes; most of them require the assembly and subsequent solution of the composite system of equations, with both the elastic and thermal terms, to allow the correct

calculation of the coupling effects. This approach, anyway, may lead to ill-conditioned linear system, with poor results, and low computational efficiency.
We found more suitable for our intent and for practical applications of the resulting code, to treat the elastic and the thermal system separately, with different approaches for time integration.
The thermal problem is in fact solved with an implicit backward Euler scheme, and the elastic problem is solved by an explicit, *leap-frog* scheme.
While the former method is unconditionally stable, the latter must satisfy the Courant-Friedrichs-Levy (CFL) condition on the time step:

$$\Delta t \leq \zeta \frac{\Delta x_{min}}{c_{max}} \tag{39}$$

where Δx_{min} is the minimum spacing in the spectral grid and c_{max} is the maximum wave propagation velocity; finally $\zeta \in (0,1)$ is a stability parameter.
The Δx_{min} in the LGL points distribution corresponds to the nodes near the border of the subdomain, where the grid spacing is proportional to L/N^2, L being the characteristic length and N the polynomial order.
For the heat transfer problem, knowing the temperature vector θ_n at time $t_n = n\Delta t$, and the vectors for the external heat sources and the elastic coupling effect at time t_{n+1} ($\mathbf{f}^{\theta\theta}_{n+1}$, $\mathbf{f}^{\theta m}_{n+1}$) it is possible to calculate the vector of the temperature θ_{n+1} at time t_{n+1} by solving:

$$\left(\frac{1}{\Delta t}\mathbf{C}^{\theta\theta} + \mathbf{K}^{\theta\theta}\right)\theta_{n+1} = \frac{1}{\Delta t}\mathbf{C}^{\theta\theta}\theta_n + \mathbf{f}^{\theta\theta}_{n+1} + \mathbf{f}^{\theta m}_{n+1} \tag{40}$$

For the elastodynamic problem, knowing the the displacements $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$, the external forces $\mathbf{f}^{mm}_{n+1}$ and the loads due to the thermal coupling $\mathbf{f}^{m\theta}_{n+1}$, it is possible, without solving any linear system (inertia matrix is diagonal), to calculate the displacements $\mathbf{u}_{n+2}$:

$$\mathbf{u}_{n+2} = \Delta t^2 \left(\mathbf{M}^{mm}\right)^{-1}\left(\mathbf{f}^{mm}_{n+1} + \mathbf{f}^{m\theta}_{n+1} - \mathbf{K}^{mm}\mathbf{u}_{n+1}\right) + 2\mathbf{u}_{n+1} - \mathbf{u}_n \tag{41}$$

Due to the different order of time derivation and to a suitable choice of the operation sequence, it is possible to assure the correct coupling between the elastic and thermal problem. Here the operation sequence is described, supposing $\mathbf{u}_{n+1}$ and θ_n known:

1. Calculate $\mathbf{f}^{\theta m}_{n+1}$ based on the displacements $\mathbf{u}_{n+1}$
2. Calculate $\mathbf{f}^{\theta\theta}_{n+1}$ not dependent by the temperature or displacement value
3. Solve the linear system and calculate θ_{n+1}
4. Calculate $\mathbf{f}^{m\theta}_{n+1}$ based on the temperatures θ_{n+1}
5. Calculate $\mathbf{f}^{mm}_{n+1}$ not dependent by the temperature or displacement value
6. Calculate $\mathbf{u}_{n+2}$
7. Print results if necessary, update time and go back to step 1

In our view, this method is the best compromise, allowing an exact treatment of the coupling, and a good computational efficiency. The elastic system is solved very fast in a full explicit way, and the time step is not excessively small. The fact that the thermal system has to be solved implicitly is a minor drawback: an explicit method would imply another limit for the allowable time step, and it appeares in many cases that this is more severe than the limit due to the elastic problem, with excessively long computation time.
The proposed method has been implemented in the framework of the spectral element library under development at CRS4, therefore inheriting from the experience gained in the implementation and use of codes designed for other physical applications.

6 Applications

The effect of thermal stresses may be important in various engineering problems. Thermal stresses are normally present in every structure but their importance is proportional to the value of temperature gradients and heat fluxes, and, for non stationary problems, to the time rate of the heating, and temperature variation in the structure.
It is common practice to adopt several approximations in solving this problem.
Apart from steady problems, where the heat transfer is stationary and the elastic problem is static, two types of transient problems are normally treated In the first class, elasto-dynamics phenomena are by far quicker then the thermal transient, therefore in this case a transient thermal problem is solved first and a set of static elastic problems are solved at times of interest[2]. This approach is used for a wide class of heating and cooling problems in engineering. In the second class of problems a sudden heating is applied (often to model some kind of explosion) and the elasto-dynamic problem is studied; the thermal problem is not solved at all, and temperature distribution is kept constant, since the characteristic time scales of its variation are assumed to be orders of magnitude higher than those involved in the elastic phenomena [1].
Obviously, the method developed may be used efficiently in solving the above classes of problems, with none or few modifications to the approach depicted above, to reach the maximum efficiency. Anyway this method is best used in a wide class of engineering problems in which these simplifications are not allowable: among the others, structural vibrations in satellites, impacts of projectiles, laser heating and welding, forging of metallic alloys [2][3][6].
If the material behavior may be considered elastic, due to the material properties of the common engineering alloys, the heating effect of the elastic deformation is often negligible, but this is not correct for some high technology recently developed materials [6].
As an example we present an application known as Danilovskaya's first problem, for which an analytical solution is known in literature, at least for the uncoupled problem, along with several well tested numerical solutions for the coupled case [2][7][8].
We consider an elastic half space ($x > 0$), with the boundary plane $x = 0$ subject to a sudden imposed temperature heating, after which the temperature is held constant. The bounding plane is traction free at all times and the elastic half space is constrained so that only uniaxial movement (perpendicular to the surface) are allowed. Hence, the problem is essentially one dimensional, but will be here solved in a two dimensional domain.
The domain considered is 4 units wide and 1 unit high; all the mechanical quantities may be chosen arbitrarily since non dimensional quantities will be used for the results analysis

$$t^* = \frac{\rho c c_s^2}{k} t \qquad x^* = \frac{\rho c c_s}{k} x \qquad T^* = \frac{T - T_0}{T_1 - T_0} \tag{42}$$

$$u^* = \frac{(\lambda + 2\mu)\rho c c_s}{k\alpha(3\lambda + 2\mu)(T_1 - T_0)} u \qquad \sigma^* = \frac{\sigma}{\alpha(3\lambda + 2\mu)(T_1 - T_0)} \tag{43}$$

$$c_s = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \delta = \frac{\alpha^2(3\lambda + 2\mu)^2 T_0}{\rho c(\lambda + 2\mu)} \tag{44}$$

In the previous equations c_s is the wave propagation velocity in the material; δ is a parameter describing the coupling effect: for traditional materials its value is in the range between 0.01 and 0.1 while some high performance composite material can reach values up to 0.5. A mesh of 40×10 quadrilateral elements is used and computation is performed using different spectral degrees.
In the graph of Fig.1 the time history of temperature and displacement for a point placed at $x^* = 1.0$ are compared with the analytical solution for the uncoupled problem. It appears how the numerical solution rapidly converges to the analytical exact solution increasing the spectral degree. The plots of Fig.2 show the results in the model at $t^* = 1.0$. The plots of Fig.3 describe the influence of the mechanical coupling: the same problem has been analyzed using different values of δ (the case with $\delta = 1.0$ does not correspond to any known material, and is presented in order to clarify the trend in the physical behavior). Several other applications are envisaged, linked to the activity of the group and will be presented in future works.

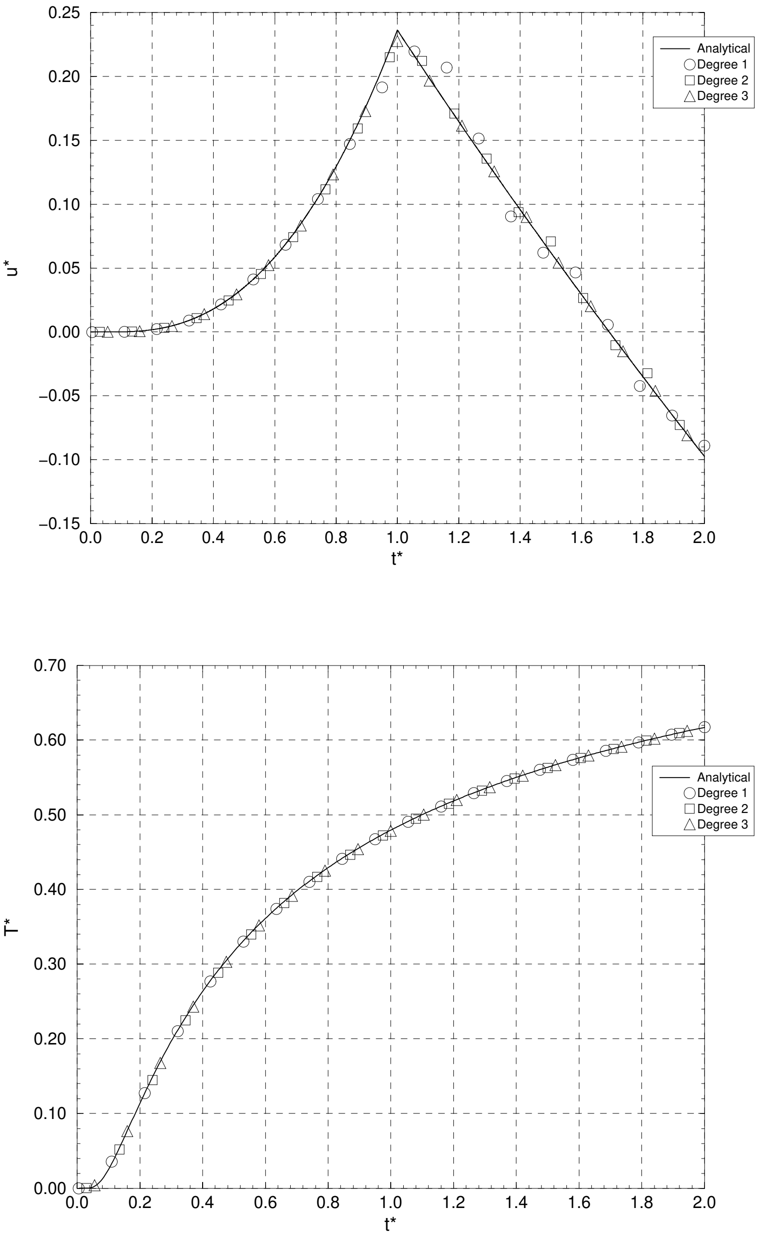
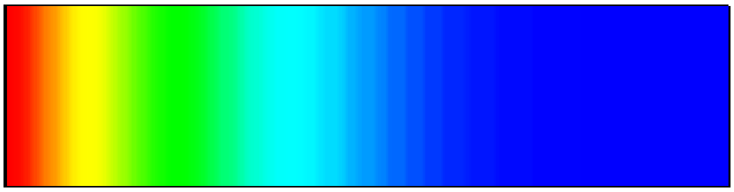
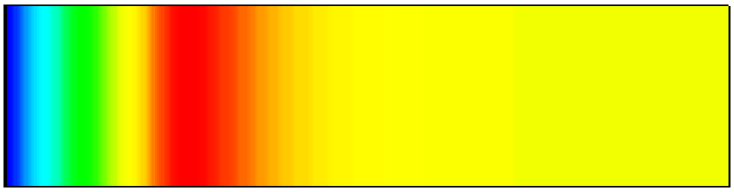


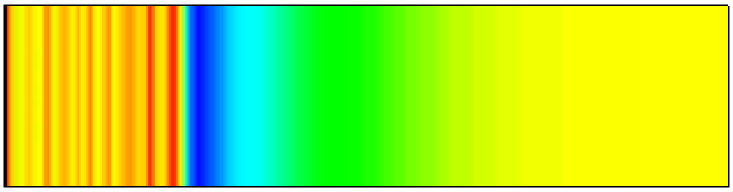
Figure 1: Comparison of analytical and numerical results for the uncopled problem with different spectral degrees



Temperature T^*

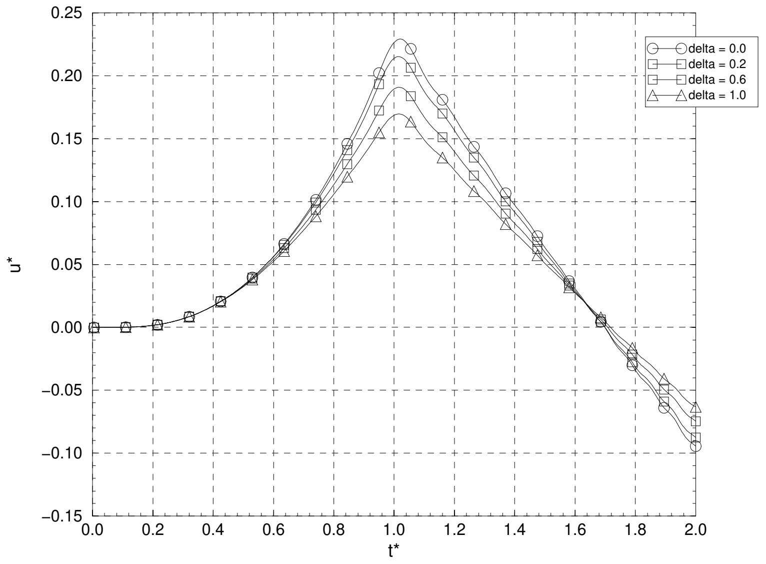


Displacement u^*



Stress σ^*

Figure 2: Plots of the results at time $t^* = 1.0$



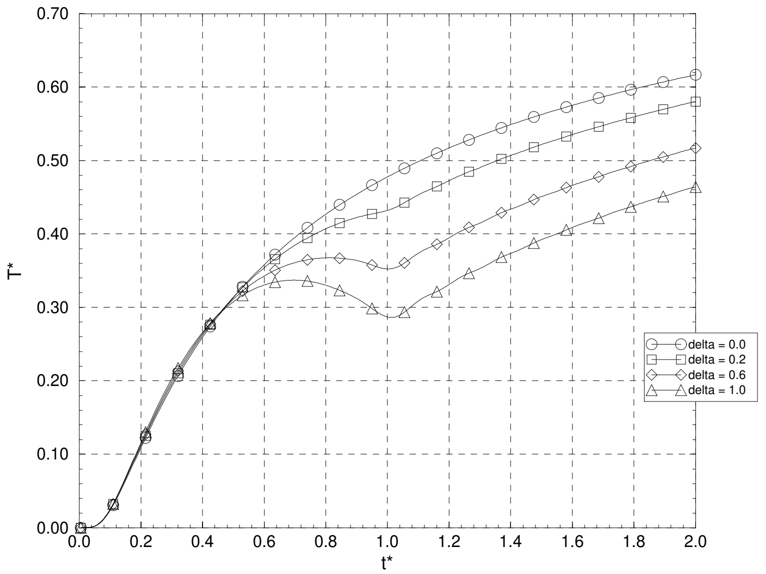


Figure 3: Comparison of numerical results for the coupled problem with different values of the coupling parameter

7 Conclusions

A method for the solution of the coupled transient thermoelastic problem has been presented. The procedure allows the solution of the fully coupled thermoelastic problem, treating the thermal and elastic problem separately and assuring the correct coupling by a suitable choice of the operation sequence. As a test case, a problem well known in literature has been presented.
The method is based on the spectral element technology under continuous development at the Solids and Structural Mechanics area at CRS4, it has all the benefits of the spectral element method, a generalization of the finite element method based on the use of high order polynomials.

References

[1] Buono S., G. Fotia, F. Maggio, and L. Massidda *Simulation of the elastic wave propagation induced by the PS beam into the TOF target at CERN*, CRS4-TECH-REP-01/57

[2] Tamma, K. K., and R. R. Namburu, *Computational approaches with applications to non-classical and classical thermomechanical problems*, Applied Mechanics Reviews, 50 (9), pp. 514-551, 1997

[3] Zhu, Y. Y., and S. Cescotto, *Transient thermal and thermomechanical analysis by mixed FEM*, Computers & Structures, 53 (2), pp. 275-304, 1994

[4] Casadei, F., E. Gabellini, G. Fotia, F. Maggio, and A. Quarteroni, *A hybrid spectral-element/finite-element method for complex wave propagation applications*, CRS4-TECH-REP-00/21, submitted to Computer methods in applied mechanics and engineering

[5] Faccioli, E., F. Maggio, A. Quarteroni, and A. Tagliani, *Spectral-domain decomposition methods for the solution of acoustic and elastic wave equations*, Geophysics, 61 (4), pp. 1160-1174, 1996

[6] Rao, D. M., and P. K. Sinha, *Finite element coupled thermostructural analysis of composite beams*, Computers & Structures, 63 (6), pp. 539-549, 1997

[7] Cannarozzi, A. A., and F. Ubertini, *Un approccio variazionale misto in dinamica termoelastica*, AIMETA - XIII Convegno Italiano di Meccanica Computazionale, Atti, pp. 682-699, 2000

[8] Danilovskaya, V.I., *Thermal stresses in an elastic half space arising after sudden heating of its boundary*, Prikl. Mat. Mekh., 14, pp 316-318, 1950

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