PRELIMINARY DESCRIPTION OF A HILLSLOPE-STORAGE BOUSSINESQ MODEL FOR SUBSURFACE AND OVERLAND FLOW

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1 Introduction

The Boussinesq model for subsurface flow in an idealized sloping aquifer has recently been extended to hillslopes of arbitrary geometry by incorporating the width function $w(x)$ into the governing equation, where x is the distance along the length of the hillslope. The resulting mathematical model can be simplified if three higher order terms containing $(1/w)w'(x)$ are dropped. In this preliminary report we describe the model along with some characteristic hillslopes that will be used to test it and develop it further.

2 Hillslope-storage Boussinesq model

2.1 Motivation

The Boussinesq equation

$$f\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[kh(\cos\alpha\frac{\partial h}{\partial x} + \sin\alpha)\right] + N \tag{1}$$

is commonly used to model subsurface flow in a sloping unconfined aquifer underlain by an impermeable layer [*Childs* 1971] (Figure 1). In this equation f is the drainable porosity, $h = h(x,t)$ is the depth of the aquifer (or height of the water table) measured perpendicular to the bedrock, t is time, x is the distance along the hillslope taken parallel to the impermeable layer, k is the hydraulic conductivity, α is the slope angle, and N is an effective recharge rate or source/sink term. Equation (1) has obvious appeal because it is one-dimensional and it can be solved analytically for a wide variety of conditions [*Serrano* 1995], in particular for the drainage-only case ($N = 0$) [*Brutsaert* 1994].

There is much interest in current hydrological research to develop simple yet physically realistic models valid at the catchment scale, focusing on the subcatchment or hillslope as a fundamental unit or building block. To this end a storage-based version of the Boussinesq model has recently been proposed [*Troch et al.* 2001], wherein, following a concept introduced by *Fan and Bras* [*1998*] for the kinematic wave model, the classical Boussinesq equation for idealized straight hillslopes is generalized by incorporating the width function $w(x)$ and introducing the subsurface water storage $S(x,t) = fwh$ as the dependent variable in the model. The resulting "hillslope-storage" Boussinesq model accomodates not only arbitrary plan curvature via $w(x)$, but also arbitrary profile shape by treating the width-averaged soil depth D as spatially variable in the x direction in the definition of the maximum subsurface water storage $S_c(x) = fD(x)w(x)$ (Figure 2). Thus the general features of a hillslope's plan geometry and terrain and bedrock shape, as derived for example from spatial analysis based on soil and digital elevation maps, can be accounted for. When solved numerically, spatial (and temporal) variability in recharge, boundary conditions, and conductivity are also readily handled.

The model can be used to simulate subsurface flow and storage dynamics on realistic hillslopes, and, via S_c, the surface saturation response activated by the saturation excess mechanism of runoff generation. Outflow hydrographs at the hillslope outlet or seepage face are easily partitioned between subsurface and overland flow contributions. We remark that the second

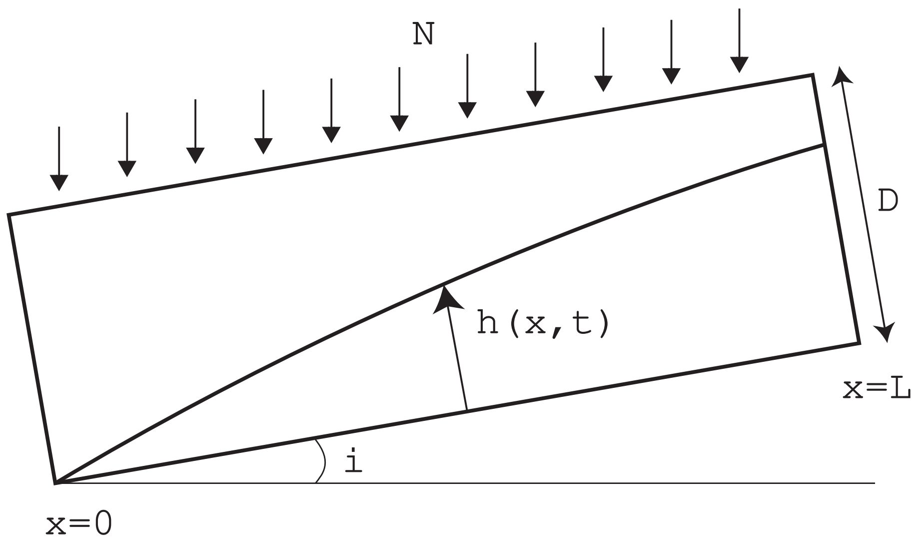

Figure 1: Schematic of a simplified straight hillslope representing a sloping unconfined aquifer underlain by an impermeable layer.

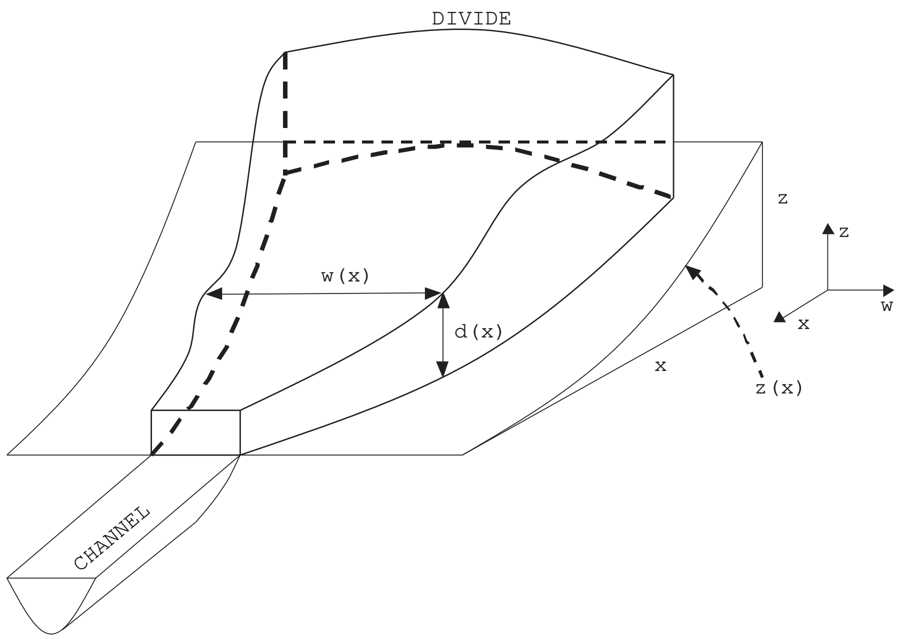

Figure 2: Schematic of a more general three-dimensional hillslope.

mechanism for generating surface runoff — infiltration excess — can only be implicitly accounted for, in the absence of an unsaturated zone component in the Boussinesq model, by considering the recharge term N as an "effective" or actual infiltration rate and not as the potential rate represented by the rainfall amount. Addition of the source/sink term N to equation (1) extends the range of applicability of the Boussinesq model from drainage studies to storm-interstorm simulations [*Verhoest and Troch* 2000].

2.2 Derivation

Combining the storage-based continuity equation

$$\frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial x} + Nw \tag{2}$$

with Darcy's law for a hillslope with width function $w(x)$

$$Q = qw = -kh(\cos\alpha\frac{\partial h}{\partial x} + \sin\alpha)w \tag{3}$$

and substituting S/fw for h in (3) we obtain the hillslope-storage Boussinesq equation

$$f\frac{\partial S}{\partial t} = \frac{k\cos\alpha}{f}\frac{\partial}{\partial x}\left[\frac{S}{w}(\frac{\partial S}{\partial x} - \frac{S}{w}\frac{\partial w}{\partial x})\right] + k\sin\alpha\frac{\partial S}{\partial x} + fNw \tag{4}$$

where Q is a volumetric discharge flux and q is the Darcy flux for a sloping unconfined aquifer of unit width. Expanding the second order derivative term in (4) gives

$$f\frac{\partial S}{\partial t} = \frac{k\cos\alpha}{fw}\left[(\frac{\partial S}{\partial x})^2 + S\frac{\partial^2 S}{\partial x^2} - \frac{3S}{w}\frac{\partial S}{\partial x}\frac{\partial w}{\partial x} + \frac{2S^2}{w^2}(\frac{\partial w}{\partial x})^2 - \frac{S^2}{w}\frac{\partial^2 w}{\partial x^2}\right] + k\sin\alpha\frac{\partial S}{\partial x} + fNw \tag{5}$$

Dropping the 3 terms containing $\partial w/\partial x$ yields the simplified form of the hillslope-storage Boussinesq model

$$f\frac{\partial S}{\partial t} = \frac{k\cos\alpha}{fw}\left[(\frac{\partial S}{\partial x})^2 + S\frac{\partial^2 S}{\partial x^2}\right] + k\sin\alpha\frac{\partial S}{\partial x} + fNw \tag{6}$$

2.3 Simplified form of the model

The simplified form of the hillslope-storage Boussinesq model given by equation (6) is a good approximation to the complete model when

$$\begin{aligned} R &= \frac{k\cos\alpha}{fw}\left[-\frac{3S}{w}\frac{\partial S}{\partial x}\frac{\partial w}{\partial x} + \frac{2S^2}{w^2}(\frac{\partial w}{\partial x})^2 - \frac{S^2}{w}\frac{\partial^2 w}{\partial x^2}\right] \\ &= -\frac{kS\cos\alpha}{f}\left[\frac{3}{w^2}\frac{\partial S}{\partial x}\frac{\partial w}{\partial x} - \frac{2S}{w^3}(\frac{\partial w}{\partial x})^2 + \frac{S}{w^2}\frac{\partial^2 w}{\partial x^2}\right] \end{aligned} \tag{7}$$

is small relative to the other terms that appear in equation (5). Applying the chain rule reduces this "residual" term to

$$R = -\frac{kS\cos\alpha}{f}\left[3\frac{\partial S}{\partial x}(\frac{1}{w^2}\frac{\partial w}{\partial x}) + S\frac{\partial}{\partial x}(\frac{1}{w^2}\frac{\partial w}{\partial x})\right] \tag{8}$$

We seek therefore a functional form for $w(x)$ such that $(1/w^2)(dw/dx)$ and its derivative are "very small". This condition is trivially satisfied for straight hillslopes ($w(x)$ constant), and indeed for this case (and taking unit width) the hillslope-storage model reduces exactly to the classical Boussinesq equation (1), as expected.

For a more general solution to the problem, the family of width functions given by

$$w(x) = \frac{\lambda_1}{x + \lambda_2} \tag{9}$$

has the property $(1/w^2)(dw/dx) = -1/\lambda_1$, a constant, and so the derivative of $(1/w^2)(dw/dx)$ vanishes and we need only satisfy the condition that $-1/\lambda_1$ be very small in magnitude, or equivalently that $|\lambda_1|$ be "very large". Analysis of the behavior of this family of width functions is the topic of future work.

3 Characteristic hillslopes

A river basin is made up of interconnected hillslopes and the channel network which drains these hillslopes. Both hillslopes and channels transport water to the outlet of the basin. In order to understand the hydrological processes at the catchment scale one needs to understand the characterisic response of the hillslopes and channel network within the catchment. We describe nine characteristic hillslopes that can be used for hydrological investigations such as model intercomparisons and numerical analysis [*Paniconi et al.* 2001].

The three-dimensional shape of our characteristic hillslopes can be described analytically. Analytical descriptions have several advantages over gridded approximations of surface shape: consistent translation between quasi-3D hillslope representations (using a function describing the hillslope width, for example) and fully 3D representations; easy generation of gridded as well as triangulated surface meshes at any regular or irregular resolution (useful for example in analyzing grid size effects or in intercomparison of finite element and finite difference models).

The hillslopes are characterised by the combined curvature in the length direction (profile curvature) and the curvature in the width direction (plan curvature). This description gives three possible shapes for profile as well as plan: concave, straight, and convex. Combining plan and profile curvature leads to nine characteristic hillslopes. The equation describing the hillslope surface shape can be written as

$$z(x, y) = E + H(1 - x/L)^n + aw^2 \tag{10}$$

where

z	[L]	= elevation above a reference point
x	[L]	= distance along the length of the hillslope
w	[L]	= slope width
L	[L]	= slope length parameter
E	[L]	= elevation at point $x = L$
H	[L]	= height difference between $x = 0$ and $x = L$
n	[-]	= profile curvature parameter ($n > 0$)
a	[1/L]	= plan curvature parameter

For $n < 1$ the profile curvature curvature is convex, for $n = 1$ it is straight, and for $n > 1$ it is concave. For $a < 0$ the plan curvature is convex, for $a = 0$ it is straight, and for $a > 0$

Table 1: Parameter Values for Nine Characteristic Hillslopes ($L = 100$)

nr	profile	plan	H	n	$a(\cdot 10^{-4})$	Area
1	concave	concave	5.01	2.00	5	2496
2	concave	straight	5.01	2.00	0	5000
3	concave	convex	5.01	2.00	-5	646
4	straight	concave	5.25	1.00	5	2160
5	straight	straight	5.25	1.00	0	5000
6	straight	convex	5.25	1.00	-5	2161
7	convex	concave	8.16	0.31	5	1410
8	convex	straight	8.16	0.31	0	5000
9	convex	convex	8.16	0.31	-5	2386

it is concave. The parameters used to generate the nine characteristic hillslopes depicted in Figure 3 are listed in Table 1.

The profile curvature is important because it reflects the change in slope angle and thus controls change of velocity of mass flowing down along the slope curve. The plan curvature reflects the change in aspect angle and determines the divergence or convergence of water flow. Thus, both plan and profile determine the location of the slope divides and consequently the slope width.

A distinction can be made between three different hillslope shapes: convergent, uniform and divergent. For convergent hillslopes (concave plan shape) the slope width decreases, for uniform hillslopes (straight plan shape) it is constant, and for divergent hillslopes (convex plan shape) it increases as one moves down the slope profile. In order to find the precise location of the slope divides, a gradient descent (for convergent and uniform hillslopes) or gradient ascent (for divergent hillslopes) is performed starting at the sides of the hillslope. The hillslopes are constructed in such a way that divergent hillslopes are widest at the outlet, convergent hillslopes are widest at the crest, and uniform hillslopes have a constant slope width. This is illustrated in Figure 4, which depicts a top view (view of the xy-plane) with contour lines and slope divides.

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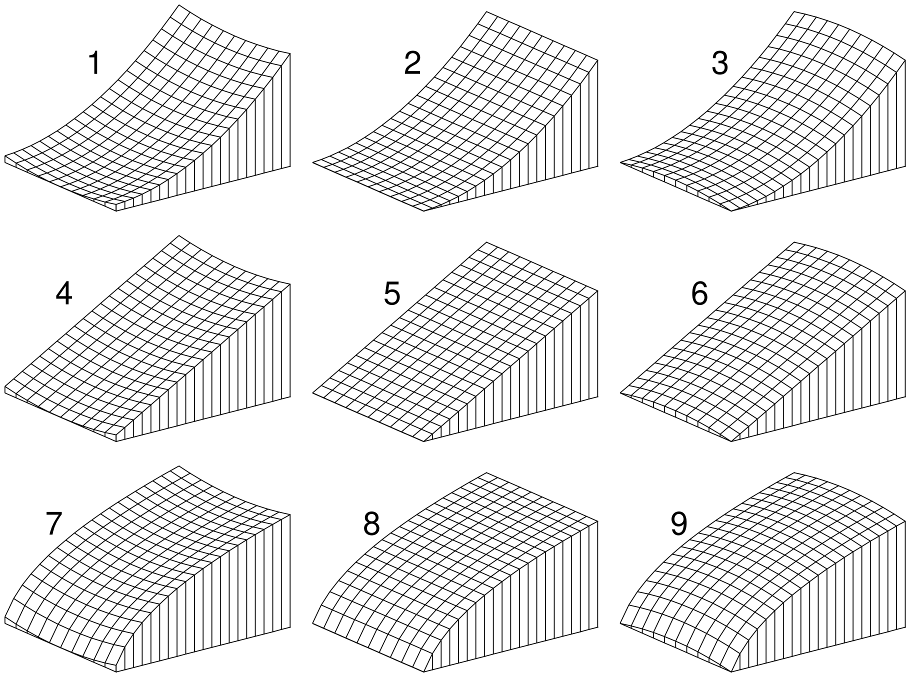

Figure 3: The nine characteristic hillslopes generated from equation (10).

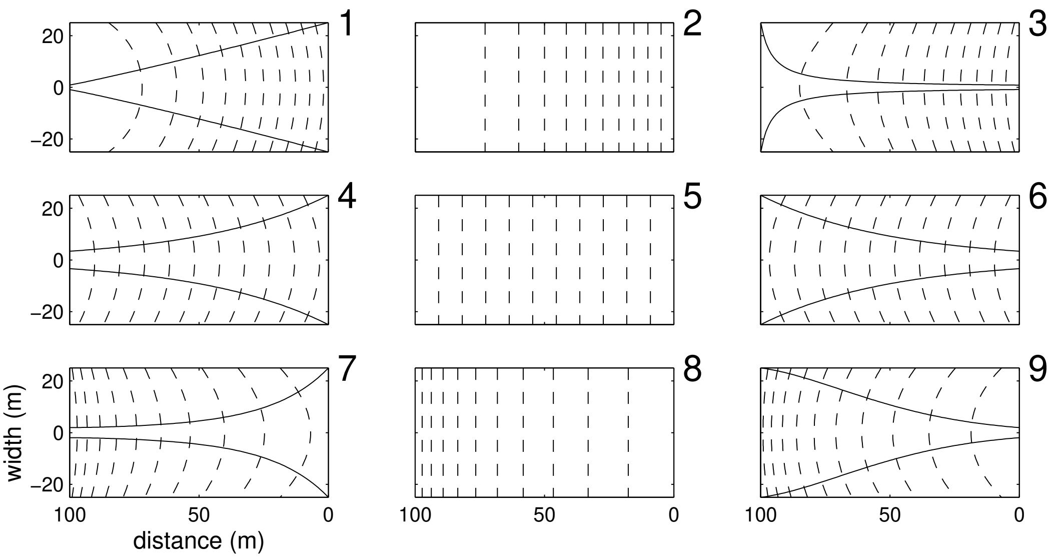

Figure 4: Contour lines and slope divides of the nine characteristic hillslopes.

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