# An optimization problem of combustion in open flow

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## 1. Introduction

Numerical optimization of the combustion is a difficult problem not only due to very complex physical-chemical processes that accompany this phenomenon but also for the complex engineering problem to produce optimal machine using combustion (power plant, jet engine and so on). A criterion of optimization reflects a compromise between, for example, the efficiency coefficient of the heat and the amount of pollution and so on. Here it's necessary to take into account existence of unsteady regime of combustion that is not admissible in industrial combustion. It is a reason why we discuss in this report the optimization problem with clear criterion of optimization. It is using of external combustion for reduction of the resistance and even for generation of the thrust of a body moving with supersonic velocity. It's a significant applied problem analyzed, for example, in [1] - [4].

In these papers optimization problems were not discussed but optimization problems were widely investigated in nonreacting aerodynamics [5] - [6]. In fact in our research we used variation methods developed in aerodynamics. In this report we analyze the combustion variation problem in exact linear formulation and develop approximate approach to a nonlinear problem based on the model of heat release in the strip of flow with zero mass flux.

## 2. Resistance of a thin body in a supersonic flow with heat release

We analyze flow around a plane body with heat release $Q$ near the surface. The system of Euler equations is the following:

$$\begin{aligned}
&\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \\
&\rho u \partial u/\partial x + \rho v \partial u/\partial y = -(1/\rho)\partial P/\partial x \\
&\rho u \partial v/\partial x + \rho v \partial v/\partial y = -(1/\rho)\partial P/\partial y \\
&\rho u \partial H/\partial x + \rho v \partial H/\partial y = w(x,t) \\
&P = \rho R T
\end{aligned} \tag{1}$$

For the case of thin bodies and small heat release $Q/c_p T_\infty << 1$ this system of equations could be linearized:

$$P = P_\infty + P_1,\ \ \rho = \rho_\infty + \rho_1,\ \ T = T_\infty + T_1,\ \ \rho = \rho_\infty + \rho_1,\ \ u = u_\infty + u_1,\ \ v = v_\infty + v_1, \tag{2}$$

where index $(_1)$ denotes small $(p_1/p_\infty << 1,\ u_1/u_\infty << 1,\ v_1/v_\infty << 1,\ \rho_1/\rho_\infty << 1)$ disturbances of the flow due to influence of the body and the heat release, index $(_\infty)$ refers to the undisturbed flow.
Inserting the expressions (2) into the equations (1) and ignoring the terms of the second and higher order we have:

$$\rho_\infty \partial u_1/\partial x + u_\infty \partial \rho_1/\partial x + \rho_\infty \partial v_1/\partial y = 0 \qquad (3)$$
$$\rho_\infty u_\infty \partial u_1/\partial x = -\partial P_1/\partial x, \qquad (4)$$
$$\rho_\infty u_\infty \partial v_1/\partial x = -\partial P_1/\partial y, \qquad (5)$$
$$\rho_\infty u_\infty (c_p \partial T_1/\partial x + u_\infty \partial u_1/\partial x) = W(x,y). \qquad (6)$$

The equation of state in the differential form after linearization is as it follows:

$$\partial P_1/\partial x = T_\infty R \partial \rho_1/\partial x + \rho_\infty R \partial T_1/\partial x. \qquad (7)$$

Eliminating $\partial P_1/\partial x,\ \partial \rho_1/\partial x,\ \partial T_1/\partial x$ from the eqs. (3) - (7), after simple manipulations we will have the resulting equation:

$$(M_\infty^2 - 1)\frac{\partial(u_1/u_\infty)}{\partial x} - \frac{\partial(v_1/u_\infty)}{\partial y} = -\frac{W(x,y)}{\rho_\infty u_\infty c_p T_\infty}. \qquad (8)$$

Now we show that the linearized equation with the heat release is nonedding flow. Eq. (4) could be integrated over $x$ that results $\rho_\infty u_\infty u_1 = -P_1 + f(y)$, where $f(y) \equiv 0$ that follows from the boundary condition: at $x \to -\infty$ $P_1 = 0$ and $u_1 = 0$, hence

$$\rho_\infty u_\infty u_1 = -P_1. \qquad (9)$$

From eqs. (5) and (6) it follows that $\partial u_1/\partial y - \partial v_1/\partial x = 0$, i.e. we have the noneddy flow with the potential $\Phi$:

$$\Phi(x,y) = \Phi_\infty(x,y) + \Phi_1(x,y),\quad u_1/u_\infty = \partial\Phi_1/\partial x,\quad v_1/u_\infty = \partial\Phi_1/\partial y. \qquad (10)$$

The equation (8) takes the form:

$$\frac{\partial^2\Phi_1}{\partial x^2} - \frac{1}{\beta^2}\frac{\partial^2\Phi_1}{\partial y^2} = -\frac{1}{\beta^2}\frac{W(x,y)}{\rho_\infty u_\infty c_p T_\infty} \qquad (11)$$

where $\beta^2 = M_\infty^2 - 1$.

The equation (11) is the inhomogeneous wave equation, its solution should be reduce to quadratura. The boundary condition on the body surface, described by the expression $y = h(x)$, is $(v/u)|_{y=h(x)} = h'(x)$ (gas does not penetrate inside the body), or in linear approximation

$$(v_1 / u_\infty)|_{y=0} = (\partial\Phi_1 / \partial y)|_{y=0} = h'(x). \qquad (12)$$

The solution of the equation (11) with the boundary condition (12) is a sum of the solution of the homogeneous wave equation with the boundary condition (12) and the solution of the inhomogeneous wave equation (11) with the zero boundary condition $(\partial\Phi_1 / \partial y)|_{y=0} = 0$:

$$\Phi_1(x, y) = \Phi_{10}(x, y) + \Phi_{1H}(x, y). \qquad (13)$$

Taking into account that the flow perturbations caused by the body is expressed only behind the body the solution could be written as it follows:

$$\Phi_{10}(x, y) = (1/\beta)\ h(x - \beta\ y). \qquad (14)$$

The solution of the inhomogeneous wave equation could be deduced by the method of reflection [4] that is based on the fact that the boundary condition automatically is valid if we analyze the problem in all space with using imaginary (fictitious) heat sources mirrored from the wall heat sources. This solution is as it follows

$$\Phi_{1H}(x, y) = \frac{1}{2\beta}\int_{-\infty}^{x} d\xi \int_{y-\frac{x-\xi}{\beta}}^{y+\frac{x-\xi}{\beta}} \frac{W(\xi,\eta)}{\rho_\infty u_\infty c_p T_\infty} d\eta. \qquad (15)$$

In accordance with eq. (10)

$$\frac{u_1}{u_\infty} = -\frac{h'(x-\beta\ y)}{\beta} - \frac{1}{2\beta^2 \rho_\infty u_\infty c_p T_\infty}\int_{-\infty}^{x} d\xi \int_{-\infty}^{x} [W(\xi, \frac{x-\xi}{\beta} + y) + W(\xi, \frac{\xi - x}{\beta} + y)] d\eta$$

and

$$\frac{v_1}{u_\infty} = -h'(x-\beta\ y) - \frac{1}{2\beta^2 \rho_\infty u_\infty c_p T_\infty}\int_{-\infty}^{x} d\xi \int_{-\infty}^{x} [W(\xi, \frac{x-\xi}{\beta} + y) - W(\xi, \frac{\xi - x}{\beta} + y)] d\eta.$$

In the linear approximation the pressure coefficient of the pressure $C_d$ using eq. (9) could be written in the form

$$C_d = \frac{2(P - P_\infty)}{\rho_\infty u_\infty^2} = \frac{2P_1}{\rho_\infty u_\infty^2} = -2\frac{u_1}{u_\infty}$$

that after expressing of the velocity in terms of the potential is the following

$$C_d = 2\frac{h'(x)}{\beta} + \frac{2}{\beta^2}\int_{-\infty}^{x} \frac{W(\xi, \frac{x-\xi}{\beta})}{\rho_\infty u_\infty c_p T_\infty} d\xi . \qquad (16)$$

We denote

$$q(x) = \frac{M_\infty}{\beta}\int_{-\infty}^{x} W(\xi, \frac{x-\xi}{\beta}) d\xi$$

the density of the integral amount of heat releasing along the characteristic arriving in the point with the coordinate $(x,0)$ and having the angle of inclination to the axis $x$ equal to $\alpha = \arcsin M_\infty^{-1}$. In this case the amount of releasing heat in the space between two characteristics arriving in point $x$ and $x + dx$ is equal to $q(x)dx / M_\infty$ and the amount of heat $Q(x)$ that is released upstream of the characteristic arriving in the point $(x,0)$ is equal to

$$Q(x) = \frac{1}{M_\infty}\int_{-\infty}^{x} q(\xi) d\xi$$

and the coefficient of the pressure on the body surface can be written as it follows

$$C_D = 2\frac{h'}{\beta} + \frac{2}{\beta}\frac{q(x)}{M_\infty \rho_\infty u_\infty c_p T_\infty} = 2\frac{h'(x)}{\beta} + \frac{2}{\beta}\frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty}. \qquad (17)$$

It means that the peculiarity of the linear solution is that the pressure in a point on the body surface is controlled only by the integral amount of the heat released on the characteristic arriving in this point and does not depend on the distribution of the heat release along the characteristic.
The final result is that the resisting force (or the thrust) $R_x$ of the body in the linear approximation is

$$R_x = \frac{\rho_\infty u_\infty^2}{2}\int_{X_i}^{X_f} C_d h'(x) dx$$

or taking into account the expression (17) it can be presented as it follows:

$$R_x = \frac{\rho_\infty u_\infty^2}{2}\int_{X_i}^{X_f} (h'^2(x) dx + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty})\ dx \qquad (18)$$

## 3. Optimization of the body profile and heat release

We use results of the previous section for defining of the coefficient of efficiency in the problem of the thrust generation by external burning in supersonic streams. We

analyze the situation when the heat release takes place only over the stern (the back part) of the body and do not take into account the impedance of the front part of the body. In this case the deduced coefficient of efficiency would be the upper estimation. In accordance with the expression (18) the longitudinal force is controlled by the profile $y = h(x)$ and the function of the heat release $Q(x)$. It is natural to introduce a limit on the value of the maximal heat release intensity $\alpha \le \alpha_{\max} = [dQ/(\rho_\infty u_\infty c_p T_\infty\ dx]_{\max}$. In our case this limit is defined by the condition of applicability of the linear theory $\alpha_{\max} << 1$. We assume that the beginning of heat release takes place at the beginning of the back part (the stern profile) of the body, Fig.2. We formulate the variation problem as it follows: between functions $Q(x)$ describing the heat release that satisfy the condition $0 \le Q' \le Q'_{\max}$ and functions $h(x)$ describing the profile of the back part of the body with boundary conditions in $x_i = 0$ and $h(x_f) = 0$ we must find such functions that result maximal coefficient of efficiency $\eta = R_x u_\infty / Q$. It is significant to stress that we do not assume any additional limitations (the body thickness, volume and so on) as they could only to reduce the coefficient of efficiency $\eta$.

The condition $0 \le Q' \le Q'_{\max}$ is equivalent to the equation

$$Q'(x)(Q'_{\max} - Q'(x)) = f^2(x) \tag{19}$$

where $f(x)$ is a new real-valued function. So the problem of maximization of the functional (18) with the differential condition (19) is reduced [2, 3] to maximization of the functional

$$I = \int_{X_i}^{X_f} F\,dx \text{ with } F = [h'(x)]^2 + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} + \mu(x)[Q'(x)(Q'_{\max} - Q'(x)) - f^2(x)] \tag{20}$$

where $\mu(x)$ is the Lagrange multiplier. Optimal functions $h(x)$, $Q(x)$ and f(x) is defined from the Euler equation of the variation problem:

$$F_h - \frac{d}{dx}F_{h'} = 0,$$
$$F_Q - \frac{d}{dx}F_{Q'} = 0,$$
$$F_f = 0.$$

which taking into account (20) have the form:

$$2h''(x) + \frac{Q''(x)}{\rho_\infty u_\infty c_p T_\infty} = 0, \tag{21}$$

$$\frac{h''(x)}{\rho_\infty u_\infty c_p T_\infty} + \frac{d}{dx}[\mu(x)(Q'_{\max} - 2Q'(x))] = 0, \tag{22}$$

$$\mu\ f = 0. \tag{23}$$

From eq. (23) it follows that the external $Q(x)$ could consist of arcs corresponding to $\mu = 0$ and $f = 0$. Along the former arc with $\mu = 0$, the strict inequality $0 < Q'(x), Q'_{\max}$ is valid. In this case from eq. (22) follows that $h' = const$ and from eq. (21) that $Q'(x) = const$. Along the latter arc with $f = 0$ either $Q'(x) = 0$ or $Q'(x) = Q'_{\max}$. In this case from eq. (21) follows that here also $h'(x) = const$. It means that optimal function $Q(x)$ consist of the linear distributions and the optimal profile consist of the straight lines. The conditions in the points of junction are the following:

$$\Delta[F - h'F_{h'} - Q'F_{Q'}]\delta x + \Delta F_{h'}\delta h + \Delta F_{Q'}\delta Q + \Delta F_{f'}\delta f = 0,$$

$$\Delta[h'^2(x) + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} - \mu(x)Q'^2(x)]\delta x + \Delta[2h'(x) + \frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty}]\delta h +$$

$$\Delta[\frac{h'(x)}{\rho_\infty u_\infty c_p T_\infty} + \mu(x)(Q'_{\max} - 2Q'(x))]\delta Q = 0,$$

where a symbol $\Delta$ denotes the difference between the values to the left and right of the point of break, $\delta x$, $\delta h$, $\delta Q$ are arbitrary variations of the angular point coordinate and the value of $Q(x)$ in this point. For arbitrary angular point (as we do not impose any limitation on its coordinates) the latter equation is equivalent to the set of equations:

$$\Delta[h'^2(x) + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} - \mu(x)Q'^2(x)] = 0,$$

$$\Delta[2h'(x) + \frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty}] = 0,$$

$$\Delta[\frac{h'(x)}{\rho_\infty u_\infty c_p T_\infty} + \mu(x)(Q'_{\max} - 2Q'(x))] = 0.$$

Hence it follows that $\Delta h' = 0$ and $\Delta Q = 0$, i. e. angular points are impossible and the optimal heat release has constant intensity $Q' = const$.

And lastly at variation of $x_i$ and $x_f$ (the coordinates of the initial and final sections of the sternpost of the body where the heat release takes place) the transversability condition must be fulfilled that is the following

$$[(F - h'F_{h'} - Q'F_{Q'})\delta x + F_{h'}\delta h + F_{Q'}\delta Q]_i^f = 0.$$

This condition for the case of arbitrary variations of $h(x_i)$, $x_f$ and $Q(x_f)$ has the form

$$h'^2(x) + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} - \mu(x)Q'^2(x) = 0,$$

$$2h'(x) + \frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty} = 0,$$

$$\frac{h'(x)}{\rho_\infty u_\infty c_p T_\infty} + \mu(x)(Q'_{\max} - 2Q'(x)) = 0,$$

where $h'$ and $Q'$ are the optimal inclination of the profile and the optimal heat release intensity. The first two equations defines the Lagrange multiplier $\mu = -0.5\rho_\infty u_\infty c_p T_\infty$. Then from the last equation it follows that $Q' = Q'_{\max}$, $h' = \dfrac{Q'_{\max}}{2\rho_\infty u_\infty c_p T_\infty}$. The optimal height of the body sternpost is controlled by the amount of the released heat $\tau = \dfrac{Q}{2\rho_\infty u_\infty c_p T_\infty}$ and the maximal coefficient of efficiency depends on the gas adiabatic exponent $\kappa = c_p / c_v$:

$$\eta = (\kappa - 1)\frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{\alpha_{\max}}{4}.$$

This expression presents also the similarity criterion: two bodies that are flowed over by gases with different adiabatic exponents and the Max numbers have the same coefficient of efficiency when

$$(\kappa_1 - 1)\frac{M_1^2}{\sqrt{M_2^2 - 1}}\alpha_{\max}^1 = (\kappa_2 - 1)\frac{M_2^2}{\sqrt{M_2^2 - 1}}\alpha_{\max}^2.$$

Fig. 3 demonstrates values of $\eta$ as a function of the maximum heat release intensities $\alpha_{\max}$ for different flow Mach numbers $M_\infty$ ($\kappa = 1.4$).
This results demonstrates that in order to generate the thrust, the body must have the wave resistance and some part of the heat is used for overcoming of this resistance. The optimal body is characterized by the wave resistance equal to ($L = x_f - x_i$ is the length of the body sternpost)

$$R_x = \frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{\alpha_{\max}}{4}L$$

that is a half of the force connected with heat release.
Without heat release minimal wave resistance in the supersonic flow equal to zero and have the infinitely thin plate and the Buzeman biplane. So for the Buzeman biplane our estimation of the coefficient of efficiency gives two times higher values.

# 4. Optimization of the body profile for given heat release in the flow

Actual heat release that takes place due to combustion at injection of a fuel into the air flow is controlled by several physical-chemical processes: turbulent mixing and molecular mixing, chemical kinetics and small-scale coupling between these processes. We assume that heat release does not depend on the body profile, i. e. we assume that the function $Q(x)$ fixed at the body profile variations are known. The variation problem formulation is the following: to find the profile $h(x)$ resulting minimal wave resistance or maximal thrust. In this problem we allow for heat release upstream of the body.

We introduce the characteristic length $L_q$ where takes place complete heat release and present the function of the heat release as it follows:

$$Q(x) = \begin{cases} 0, \ \ x < X_q, \\ \tilde{Q}(x), \ \ X_q \le x \le X_q + L_{q,} \\ 0, \ \ x > X_q + L_{q.} \end{cases} \tag{24}$$

The resistance of the body per unit of the width is the following

$$R_x = \frac{\rho_\infty u_\infty^2}{\beta} \int_{X_i}^{X_f} (h'(x) + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} + C_f)dx \tag{25}$$

where $C_f$ is the coefficient of restriction that is assumed to be constant.

Formulation of the problem is the following: between functions $h(x)$ describing the profile of a body with fixed length $X_f - X_i = L$ and boundary conditions on the beginning and the end of the body $h(X_i) = h(X_f) = 0$ we must find such profile that minimize the integral in eq. (25) where Q(x) is described by eq. (24). The body can has additional restriction such as the maximal thickness $H$ and the area $S$:

$$S = \int_{X_i}^{X_f} h(x) \ dx \, . \tag{26}$$

This problem with the isoperimetric condition (26) is equivalent to defining of the extreme of the functional

$$R_x = \frac{\rho_\infty u_\infty^2}{\beta} \int_{X_i}^{X_f} F \ dx \tag{27}$$

where $F = h'^2(x) + \frac{Q'(x)h'(x)}{\rho_\infty u_\infty c_p T_\infty} + C_f + \lambda h'(x)$ and $\lambda$ is the Lagrange multiplier.

The Euler equation of the variation problem in this case is the following:

$$\lambda - 2h''(x) + \frac{Q''(x)}{2\rho_\infty u_\infty c_p T_\infty} = 0$$

and hence after integration

$$h(x) = -\frac{Q(x)}{2\rho_\infty u_\infty c_p T_\infty} - \frac{\lambda x^2}{4} + ax + b\,, \qquad (28)$$

where $a$ and $b$ are constant which should be defined from the boundary conditions.

In the general case the optimal profile consists of several arcs described by eq. (28). In the junction points the conditions are the following:

$$\Delta[-h'^2(x) + \lambda\, h(x)]\delta x + \Delta[2h'(x) + \frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty}]\delta h = 0$$

where the symbol $\Delta$ denotes the difference between the values in brackets to the left and right of the point of break. Without any restrictions on the position on the angular points we have instead

$$\Delta[h'^2(x)] = 0\,, \qquad \Delta[2h'(x) + \frac{Q'(x)}{\rho_\infty u_\infty c_p T_\infty}] = 0\,. \qquad (29)$$

If the given thickness of the body is achieved in a junction point it would be sufficient the condition

$$\Delta[h'^2(x)] = 0\,. \qquad (30)$$

And at last in the case when the coordinates of the beginning and the end of the body are moved with respect to the zone of heat release the transversability condition must be valid. As we assumed that $h(X_i) = h(X_f) = 0$ the transversability condition is the following:

$$[-h'^2(X_i) + C_f]\delta X_i - [-h'^2(X_f)]\delta X_f = 0\,. \qquad (31)$$

The Lagrange multiplier $\lambda$ is found from the isoperimetric condition (26).

We analyze first the optimal profile of the body without external heat release and we will compare the restriction of this body with the restriction of the optimal body with heat release. For given length and thickness of the body we will analyze particular cases corresponding to different values of the area of the isoperimetric condition. It would be more convenient to set the value of the Lagrange multiplier $\lambda$ or the geometric parameters of the body and to define corresponding values of the area, the body profile and the restriction.

a) Let $C_f = 0$ and $\lambda = 0$, the length of the body L and the thickness H are given. In this case in accordance with (28) the profile consist of straight lines. From the transversability condition (31) (as at $L = const$ $\delta X_i = \delta X_f$) it follows that $[h'(X_i)]^2 = [h'(X_f)]^2$ and the optimal body in this case is the rhomb with given thickness in the middle of the bisecant, with the area of the cross section $S = 0.5LH$ and with the wave resistance $R_x = \frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}} \frac{H^2}{L}$.

b) Let us choose such value of $\lambda$ that in the section with maximum thickness the profile would have zero derivative< i. e. in the section with $h(x) = H/2$ $dh/dx = 0$. In this case the optimal profile is described by one curve of the second order (28) and taking into account (31) we have $h(x) = 2H\frac{x}{L}(1 - \frac{x}{L})$, $\lambda = \frac{8H}{L^2}$, $S = \frac{2}{3}HL$, $R_x = \frac{4}{3}\frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{H^2}{L}$. This smooth profile has the area and the restriction that is $4/3$ times greater than has the rhomb type body.

c) In the case $-\infty < \lambda < -\frac{8H}{L^2}$ the optimal body consist of the arcs 1-2, 3-4 and the straight line 2-3, and in accordance with (30) in the points 2 and 3 there is no break, i. e. there is no the derivative discontinuity of $h'(x)$. If point 2 has the coordinate $0.5\xi$ where $\xi << 1$ the optimal body is describes by the formulas (the body is symmetrical by $x = 0.5L$):

$$h(x) = \begin{cases} 2H\frac{x}{\xi L}(1 - \frac{x}{\xi L}), & 0 \le x \le 0.5\xi \\ \frac{H}{2}, & 0.5\xi \le x \le 0.5 \end{cases}$$

and here $\lambda = \frac{8H}{\xi^2 L^2}$, $S = (1 - \frac{\xi}{3})HL$, $R_x = \frac{4}{3\xi}\frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{H^2}{L}$.

In the case $\alpha \to 0$ the optimal body tend to the rectangle with the sides $H$ and $L$ (i. e. $S \to HL$) and $R_x \to \infty$. It is obvious that this limiting case is pure formal as the linear theory is not valid here.

d) $\lambda > 0$ and $S < 0.5HL$. It is easy to check that in this case $h(x) = \frac{\lambda x^2}{4} + (\frac{H}{L} - \frac{\lambda L}{8})x$, $0 \le x \le \frac{1}{2}$,

$$S = \frac{HL}{2} - \lambda\frac{L^3}{48}, \quad R_x = \frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}}(\frac{\lambda^2 L^3}{192} + \frac{H^2}{L}).$$

Here from the condition $h(x) \ge 0$ follows maximal possible value $\lambda = 8H/L$. For this value $dh/dx = 0$ in the points $x = 0$ and $x = L$ and $dh/dx = 2H/L$ in the point $x = 1/2$. In this case the area $S = HL/3$ that is minimal possible and the restriction of such concave body is equal to the restriction of the convex body with smooth profile in the section with maximum thickness. In the case $\lambda > 8H/L^2$ takes place the degenerated case when between points 1-2 and 4-5 the thickness

equal zero, i. e. at the area $S < HL/3$ in fact the length of the optimal profile is reduced and it is situated between the points 2 and 4 so the linear theory is not valid.

e) In accordance with the recommendations from the literature the linear theory is valid when $0.03 < H/L < 0.1$. The particular solutions from the items a)-d) are the classical solutions of the theory of optimal aerodynamics forms [5]. At fixed length of the optimal body profile does not depend on friction. But without restriction on the body length it directly controlled by friction. In the literature there are such solutions for the bodies with fixed thickness and the area of the cross-section.

f) Now we analyze the case of external combustion. Sequence and number of arcs described by eq. (28) and constituting the optimal profile depend on coordinates of beginning and end of the heat release $X_q$ and $X_q + L_q$ as well as by coordinate $x_0$ where the body has the maximum thickness $h(x_0) = H$. Here possible different situations: when heat release begins upstream from the section of the maximum thickness; when heat release takes place on the back side of the body; and when heat release begins in the section of maximal thickness. From eq. (29) for the angular point $X_q$ written for the cases $X_q > x_0$ or $X_q < x_0$ it follows that $Q'(X_q) = 0$ and $h_2'(X_q) = h_2'(X_q)$. Similar situation takes place in the angular point $X_q + L_q$. But in the case $Q'(X_q) \neq 0$ in accordance with eq. (30) beginning of heat release can be only in the section with the maximal heat release $X_q = x_0$. Finally the possible optimal configuration consists of two arcs $h_1(x)$ and $h_2(x)$ corresponding to eq. (28) and containing one angular point in $x = x_0$. Now we take a look at the case when $Q'(X_q) = Q'(X_q + L_q)$ that corresponds to physically reasonable case of smooth beginning in $x = X_q$ and smooth extinguish in $x = X_q + L$ of combustion. From the set of equations

$$\begin{cases} h_1(X_i) = 0, \;\; h_1'(X_i) = \sqrt{C_f}, \\ h_2(X_f) = 0, \;\; h_2'(X_f) = \sqrt{C_f}, \\ h_1(x_0) = H, \;\; h_2(x_0) = H, \\ h_1'^2(x_0) = h_2'^2(x_0) \\ S = \int\limits_{X_i}^{X_f} h(x) \;\, dx \end{cases}$$

corresponding this case the unknown $x_0$, $X_i$ and $X_f$ can be found and after this eq. (27) gives an opportunity to find the minimal restriction of the optimal body.

## 5. A model of heat release in the strip of flow with zero mass flux. Estimation of the heat release efficiency coefficient for generation of the thrust in the nonlinear case.

Our previous results refer to the linearized problem that is valid when $\alpha = dQ / \rho_\infty u_\infty c_p T_\infty \; dx << 1$. For estimation of the efficiency coefficient $\eta$ when the linear theory is not valid we will analyze the flow over the body taking into account strong distortions. The profile of the body and the heat release law we will take from the linear solution of the variation problem. We will analyze the flow shown in the Fig. 4 where head release is accompanied by formation of the shock wave. We assume that heat release takes place at constant pressure and analyze the limiting case of zero mass flux in the strip of flow where takes place heat release.

Now we present necessary equations. The plane 1D strip flow with heat release with constant pressure is describe by the set of equations

$$\begin{cases} dG = d(\rho u F) \\ dI = p dF \\ G d(c_p T + \dfrac{u^2}{2}) = dQ(x). \end{cases}$$

From the first equation it follows that $u = const$, i. e. the velocity in constant at isobaric heat release. Simple manipulations result the formula

$$F(x) = F_0 + \frac{\kappa - 1}{\kappa P u} Q(x)$$

that shows that area of the strip of flow is controlled only by heat release ($F_0$ is the initial area). In the limiting case when $F_0 = 0$ we have for constant intensity of heat release an expression for the derivation angle of the velocity from the body surface

$$tg\ \varphi = \frac{\kappa - 1}{\kappa P u} \frac{dQ}{dx}, \quad \text{or} \quad \alpha = \frac{P}{P_\infty} \frac{u}{u_\infty} tg\ \varphi \quad \text{where} \quad \alpha = \frac{1}{\rho_\infty u_\infty c_p T_\infty} \frac{dQ}{dx} \qquad (32)$$

Taking into account eqs. (32) the expression for the coefficient of efficiency for the configuration presented on the Fig. 4 is the following:

$$\eta = \frac{\left( \dfrac{P(\beta)}{P_\infty} - 1 \right) H}{Q} u_\infty = \frac{\kappa - 1}{\kappa} \frac{\left( \dfrac{P(\beta)}{P_\infty} - 1 \right) H}{\alpha} \frac{\sin\ \omega}{\cos \dfrac{\omega + \beta}{2}}, \qquad (33)$$

where H in the thickness of the body, $\beta$ is the deviation angle of the flow behind the shock wave, the pressure $P$ and the flow velocity $u$ is controlled by the shock wave, the angle of the shock wave depends on the heat release intensity $\alpha$.

The geometrical parameters giving maximum value $\eta$ were found numerically for given values $\alpha$. Fig. 5 - 8 present results for $\eta$ corresponding to different Max numbers of the flow $M_\infty$ and Fig. 9 demonstrate optimal dimensionless thickness $\overline{H} = H / L$
as a function of $\alpha$. At small heat release this results are in agreement with the linear theory. Maximal value of $\eta$ (and maximum heat release) corresponds to the boundary of the flow separation. Fig. 10 shows maximal $\eta$ as a function of the flow Max number $M_\infty$.

## 6. Conclusions

1. We formulated and resolved the problem in the linear formulation of optimal heat release and body profile for minimal resistance or minimal thrust for plane symmetrical bodies at supersonic velocities. We have analyzed the isoperimetric problems aiming to find both optimal body profile and optimal heat release distribution and, at given heat release intensity, to find optimal body profile yielding maximum coefficient of efficiency.

2. We approximately analyzed the variation problem in nonlinear formulation based on the model of heat release in the strip of flow with zero mass flux and presented quantitative results.

3. Developed method could be applied to another combustion problem connected, for example, with problems of industrial combustion.

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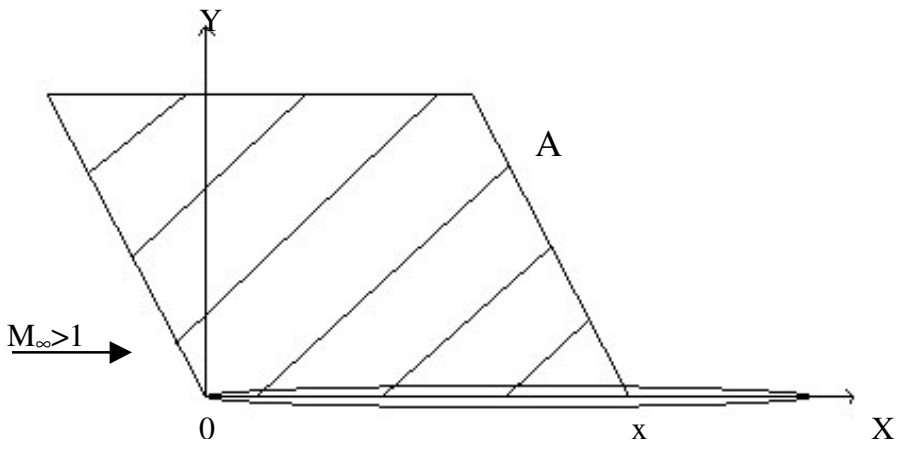



Fig.1

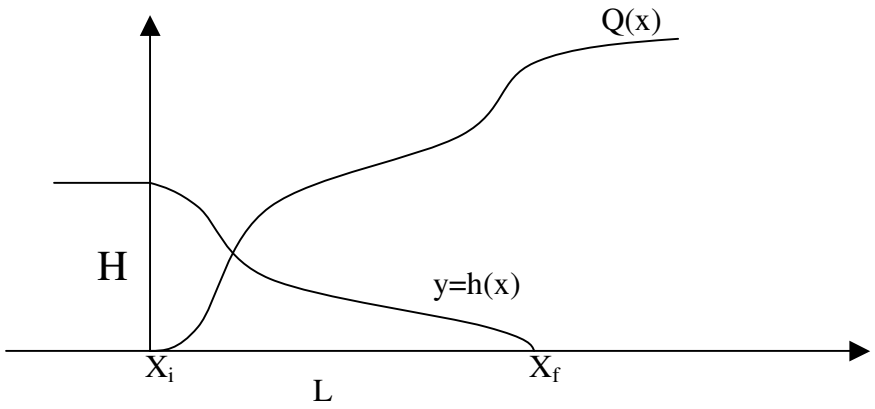



Fig.2

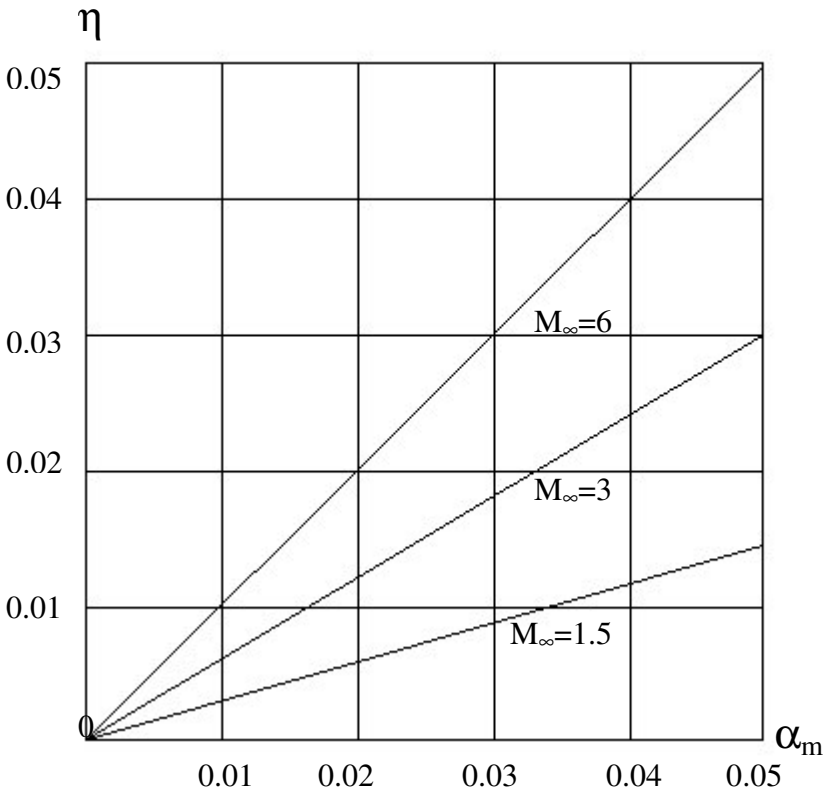

η
0.05
0.04
0.03
0.02
0.01
0
M∞=6
M∞=3
M∞=1.5
αm
0.01
0.02
0.03
0.04
0.05

Fig.3

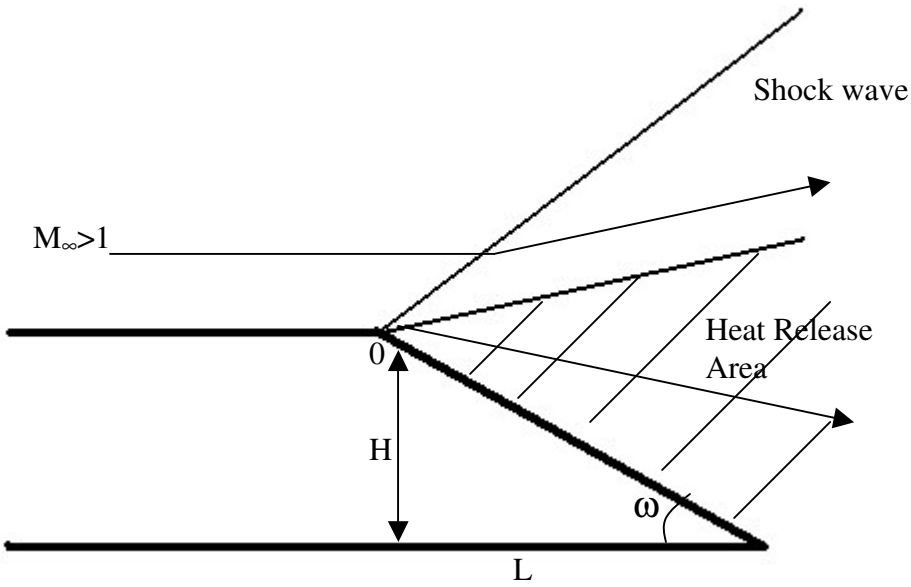

Shock wave
M∞>1
Heat Release
Area
0
H
ω
L

Fig.4

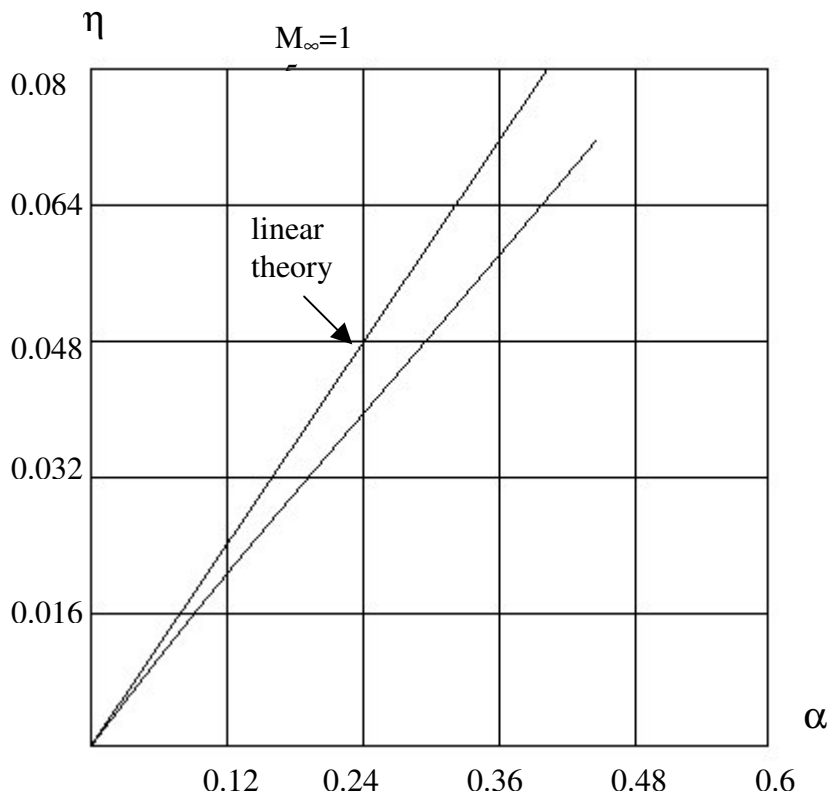

η
M∞=1
0.08
0.064
0.048
0.032
0.016
linear
theory
α
0.12
0.24
0.36
0.48
0.6

Fig.5

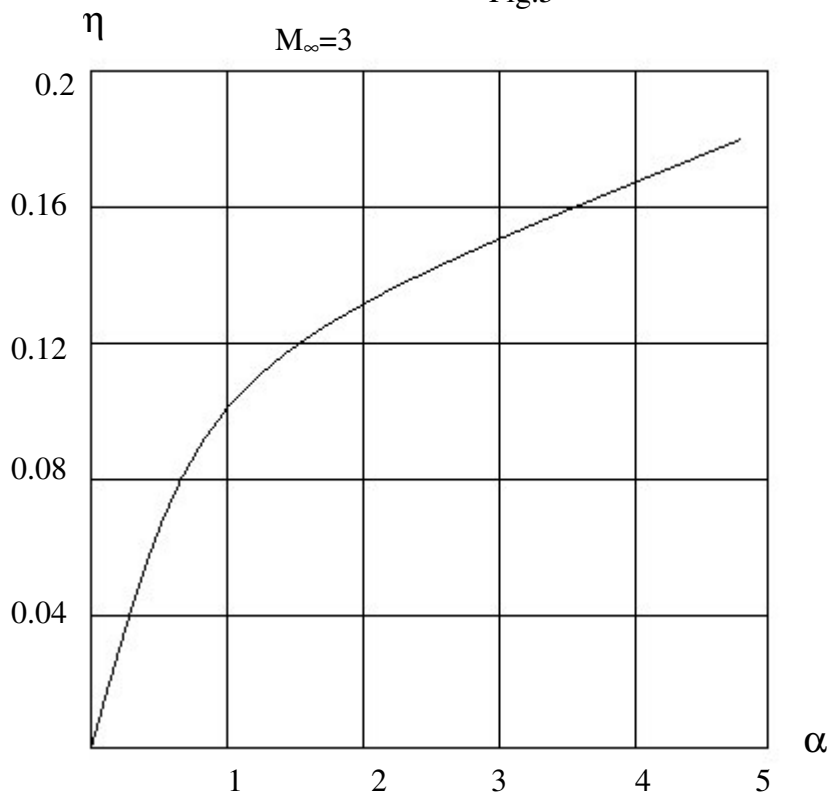

η
M∞=3
0.2
0.16
0.12
0.08
0.04
α
1
2
3
4
5

Fig.6

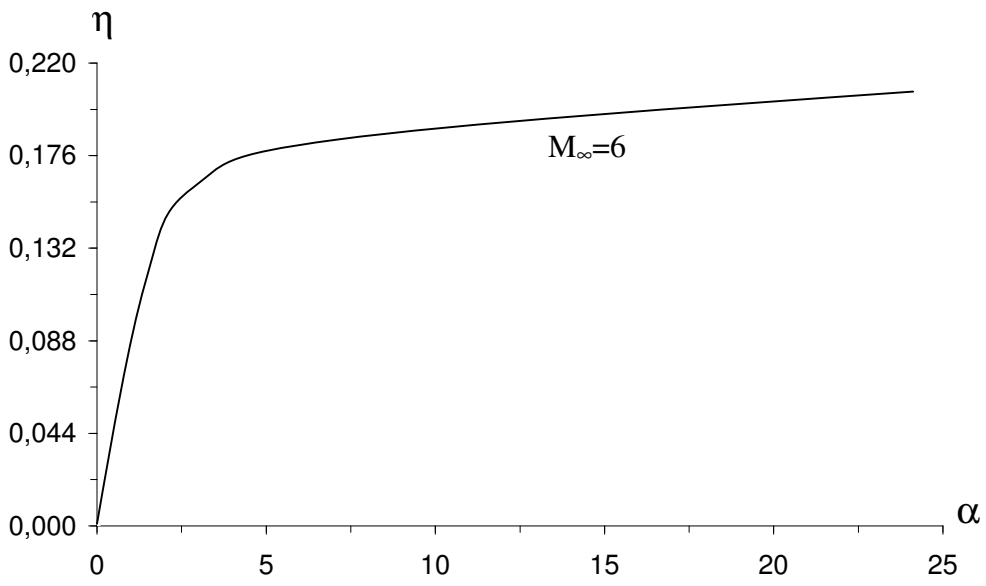
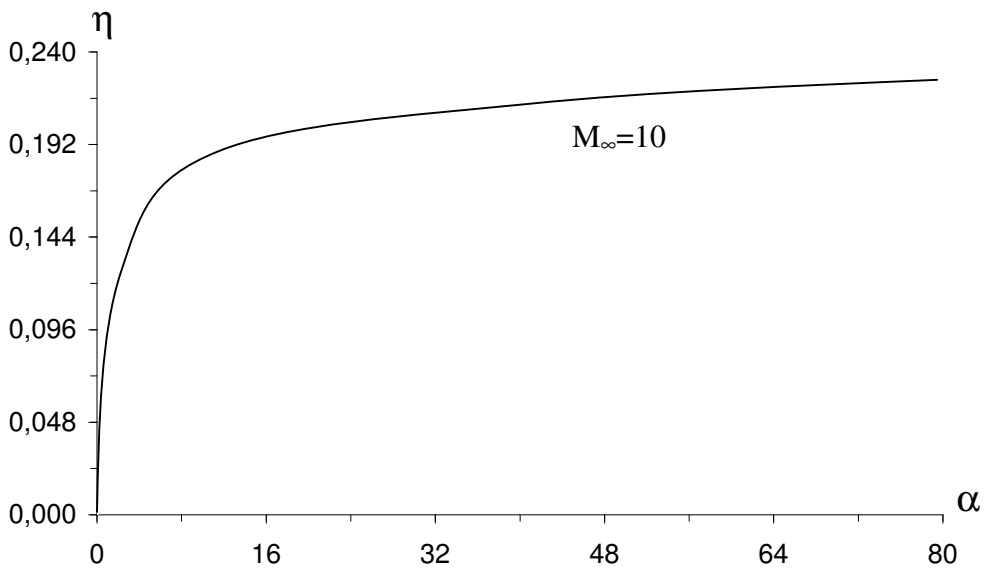

η
0,220
0,176
0,132
0,088
0,044
0,000
M∞=6
α
0
5
10
15
20
25

Fig.7


η
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0,144
0,096
0,048
0,000
M∞=10
α
0
16
32
48
64
80

Fig.8

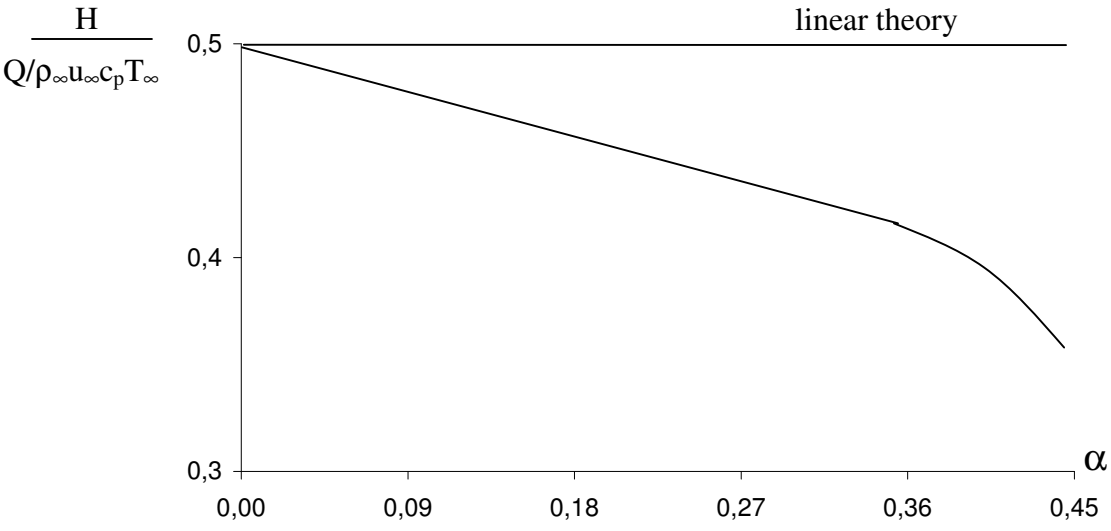
H
Q/ρ∞u∞cpT∞
linear theory
0,5
0,4
0,3
α
0,00
0,09
0,18
0,27
0,36
0,45

Fig.9

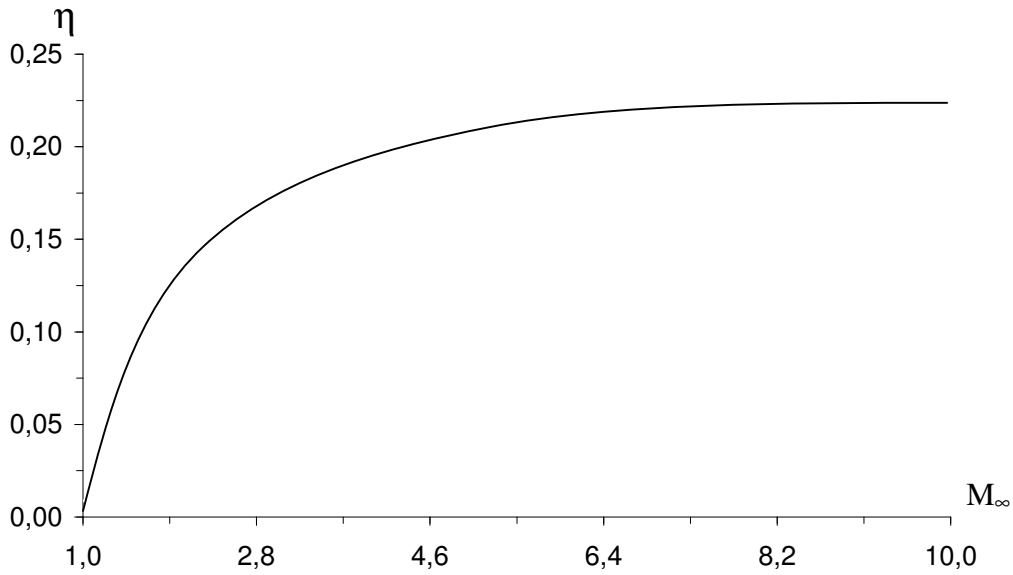
η
0,25
0,20
0,15
0,10
0,05
0,00
M∞
1,0
2,8
4,6
6,4
8,2
10,0

Fig.10