Common-Angle Migration and Oriented Waves in the Phase-Space $(\mathbf{x}, \mathbf{p})$

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From the 3D phase-shift extrapolation of individual phase-space components, we derive for prestacked data a vector relation between $\mathbf{k_h}$, the horizontal offset wavenumber, and k_z, the vertical one. While in 2D this relation depends only on the tangent of the scattering angle θ and not on the structural deep, in 3D, *in general*, this is no longer true. The resulting vector formula takes into account the orientation of the scattering plane containing the slowness vectors $\mathbf{p_s}$ and $\mathbf{p_r}$, one describing the down-going wave and the other the up-going one.

For scattering events taking place on vertical planes, we recover the expected 2D result. In this special case, we have a complete theory, first, to construct the *angle-domain, common-image* gathers, one for each scattering angle θ, and, second, to retrieve from those the *medium structure* by adequately summing over all values of θ.

Wave propagation in the phase-space $(\mathbf{x}, \mathbf{p})$

Phase-space wavefield: V, the pressure field, is not only a function of position $\mathbf{x} = (\mathbf{y}, z)^\top$ and time t, but also of $\mathbf{p} = (\mathbf{p_y}, p_z)^\top$, the wavefront *orientation* which must satisfy $||\mathbf{p}|| = n(\mathbf{x})$, the *eikonal equation*.

Example of an oriented wave in homogeneous media: Fomel's formulation (2003) leads to the following *phase-shift* equation:

$$\frac{\partial \hat{V^{\pm}}}{\partial z} = \imath k_z \ \hat{V^{\pm}} \ ; \qquad\qquad k_z = \frac{\pm \omega n^2 - \mathbf{k_y} \cdot \mathbf{p_y}}{\sqrt{n^2 - ||\mathbf{p_y}||^2}}$$

$$\hat{V^{\pm}} \left(z + \triangle z, \mathbf{k_y}, \mathbf{p_y}, \omega\right) = e^{\imath k_z \triangle z} \ \hat{V^{\pm}} \left(z, \mathbf{k_y}, \mathbf{p_y}, \omega\right)$$

The 2D time section $V_0(y,t) = \delta(y - y_0)\delta(t - t_0)$, back-propagated along the direction $\mathbf{p}/n = (\sin\alpha, \cos\alpha)^\top$, becomes at $t = 0$

$$I_\alpha \left(z, y\right) = \frac{\cos(|\alpha|)}{n} \ \delta\!\left[(y - y_0) - \frac{t_0}{n}\sin\alpha\right] \ \delta\!\left[z - \frac{t_0}{n}\cos\alpha\right] ; \ \ |\alpha| \leq \pi/2$$

3D Dispersion relation in homogeneous media: the simultaneous extrapolation of both source and receiver, for each *phase-space* components, leads to

$$k_z = \frac{\omega n^2 - \left(k_s^{(1)} p_s^{(1)} + k_s^{(2)} p_s^{(2)}\right)}{p_s^{(3)}} + \frac{\omega n^2 - \left(k_r^{(1)} p_r^{(1)} + k_r^{(2)} p_r^{(2)}\right)}{p_r^{(3)}}$$

Down-going slowness vector:

$$\mathbf{p_s} = \left(p_s^{(1)}, p_s^{(2)}, p_s^{(3)}\right)^\top = \frac{\mathbf{k_s}}{\omega}$$

Up-going slowness vector:

$$\mathbf{p_r} = \left(p_r^{(1)}, p_r^{(2)}, p_r^{(3)}\right)^\top = \frac{\mathbf{k_r}}{\omega}$$

Wave scattering of angle θ

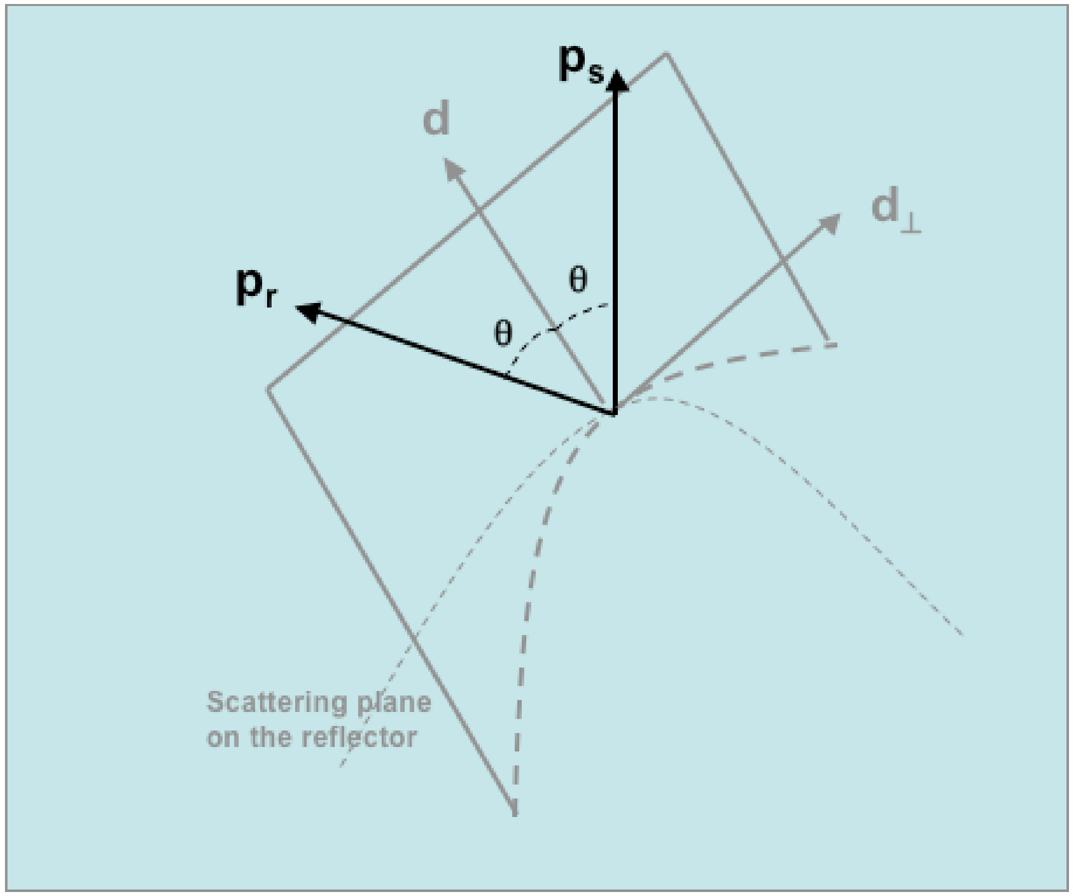

Figure 1: Both vectors $\mathbf{p_s}$ and $\mathbf{p_r}$ lie in the $(\mathbf{d}, \mathbf{d}_\perp)$-plane

The scattering plane: $\mathbf{d}$ is the normal vector to the reflecting surface at the contact point, $\| \mathbf{d} \| = 1$

while vector $\mathbf{d}_\perp$ belongs to the tangent plane at the contact point, $\| \mathbf{d}_\perp \| = 1$, $\mathbf{d} \cdot \mathbf{d}_\perp = \mathbf{0}$, so that

$$\mathbf{p_s} = n\,(\cos\theta\ \mathbf{d} + \sin\theta\ \mathbf{d}_\perp)$$

$$\mathbf{p_r} = n\,(\cos\theta\ \mathbf{d} - \sin\theta\ \mathbf{d}_\perp)$$

In addition, we define:

$$\mathbf{k}\ = \omega\,(\mathbf{p_s} + \mathbf{p_r}) = (\mathbf{k_m}, k_z)^\top$$

The scattering plane

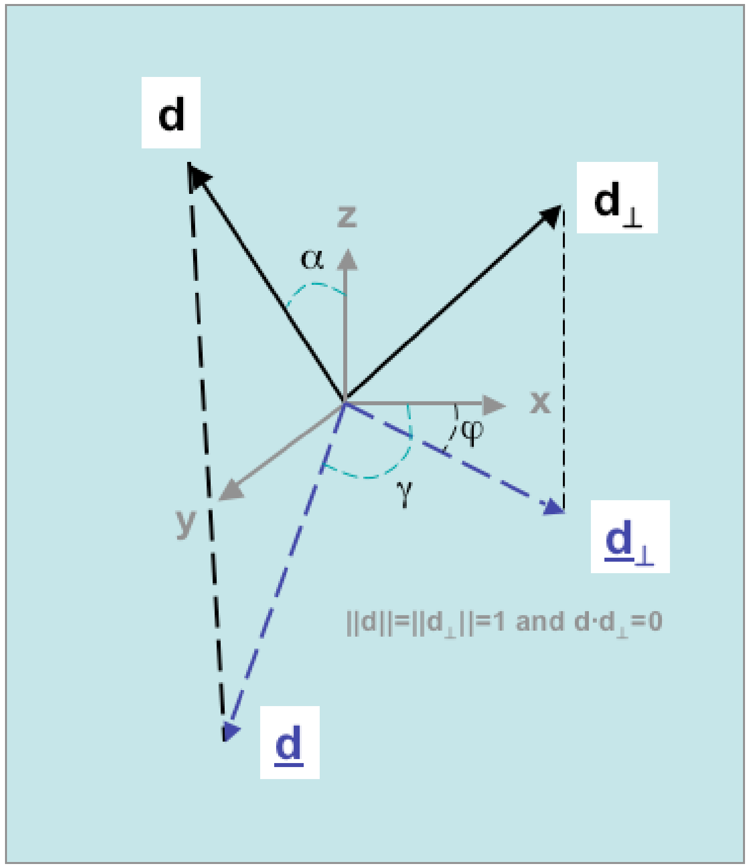

Figure 2: The plane is defined by three angles: α, γ and φ

Normal and tangent vectors: $\mathbf{d} = \left(\underline{\mathbf{d}}, d^{(3)}\right)^\top \quad \mathbf{d}_\perp = \left(\underline{\mathbf{d}}_\perp, d_\perp^{(3)}\right)^\top$

$$\underline{\mathbf{d}} = \sin\alpha \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}; \qquad d^{(3)} = \cos\alpha$$

$$\underline{\mathbf{d}}_\perp = \tfrac{1}{C}\cos\alpha \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}; \quad d_\perp^{(3)} = -\tfrac{1}{C}\sin\alpha\cos(\gamma-\varphi)$$

$$C = \sqrt{1-\sin^2\alpha\ \sin^2(\gamma-\varphi)}$$

Special case: if $\gamma = \varphi \pm m\pi$, then $\mathbf{d}$ and $\mathbf{d}_\perp$ both lie in a vertical plane and $C = 1$

Horizontal offset and midpoint wavenumbers:

$$\mathbf{k_h} = \begin{pmatrix} k_r^{(1)} - k_s^{(1)} \\ k_r^{(2)} - k_s^{(2)} \end{pmatrix} \; ; \qquad \mathbf{k_m} = \begin{pmatrix} k_r^{(1)} + k_s^{(1)} \\ k_r^{(2)} + k_s^{(2)} \end{pmatrix}$$

Dispersion relation in the midpoint-offset domain:

$$\boxed{\begin{aligned} \frac{p_s^{(3)} p_r^{(3)}}{n^2}\, k_z \;\; = \;\; & 2n\omega \cos\theta \cos\alpha - \left[\left(\cos^2\theta \; d^{(3)} \; \underline{\mathbf{d}} \; - \; \sin^2\theta \; d_\perp^{(3)} \; \mathbf{d}_\perp \right)^\top \cdot \mathbf{k_m} - \right. \\ & \left. \sin\theta \cos\theta \left(d^{(3)} \; \underline{\mathbf{d}}_\perp \; - \; d_\perp^{(3)} \; \underline{\mathbf{d}} \right)^\top \cdot \mathbf{k_h} \right] \end{aligned}}$$

Useful formulas:

$$\mathbf{k_m} = 2n \; \omega \cos\theta \; \underline{\mathbf{d}} \; ; \quad k_z = 2n \; \omega \cos\theta \cos\alpha$$

$$p_s^{(3)} p_r^{(3)} = n^2 \left[\left(d^{(3)} \; \cos\theta \right)^2 - \left(d_\perp^{(3)} \sin\theta \; \right)^2 \right]$$

$$\| \; \underline{\mathbf{d}} \; \|^2 = 1 - (d^{(3)})^2 \; ; \quad \underline{\mathbf{d}}^\top \cdot \underline{\mathbf{d}}_\perp = -d^{(3)} d_\perp^{(3)}$$

A simple scalar relation

Constraining $\mathbf{p_s}$ and $\mathbf{p_r}$ to lie in the $(\mathbf{d}, \mathbf{d}_\perp)$-plane, *prestack depth extrapolation* of individual *phase-space* components leads to the following relation

$$\boxed{\tan\theta\ k_z = -\left(d^{(3)}\ \underline{\mathbf{d}}_\perp\ -\ d_\perp^{(3)}\ \underline{\mathbf{d}}\right)^\top \cdot \mathbf{k_h}}$$

where

$$d^{(3)}\underline{\mathbf{d}}_\perp - d_\perp^{(3)}\underline{\mathbf{d}} = \frac{1}{C}\left[\cos^2\alpha \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + \sin^2\alpha\ \cos(\gamma-\varphi) \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}\right]$$

A simple vector relation

Remark: $\mathbf{k_h} = 2\omega \sin\theta \ \underline{\mathbf{d}}_{\perp}$, so that

$$\frac{k_h^{(2)}}{k_h^{(1)}} = \tan\varphi$$

Then, the relation between the vertical wavenumber k_z and the horizontal offset wavenumber $\mathbf{k_h}$ can be inverted

$$\boxed{\mathbf{k_h} = -\frac{\tan\theta \ k_z}{\sqrt{1-\sin^2\alpha \ \sin^2(\gamma-\varphi)}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}}$$

Vertical scattering planes

For $\gamma = \varphi \pm m\pi$: the relation simplifies to a form *independent* of α, the structural deep:

$$\boxed{\mathbf{k_h} = -\tan\theta\ k_z \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}}$$

In the absence of transversal structural dips: this last form is equivalent ($\varphi = 0$) to the 2D scalar one suggested by Stolt and Weglein (1985)

$$\boxed{k_h = -\tan\theta\ k_z}$$

Fourier domain

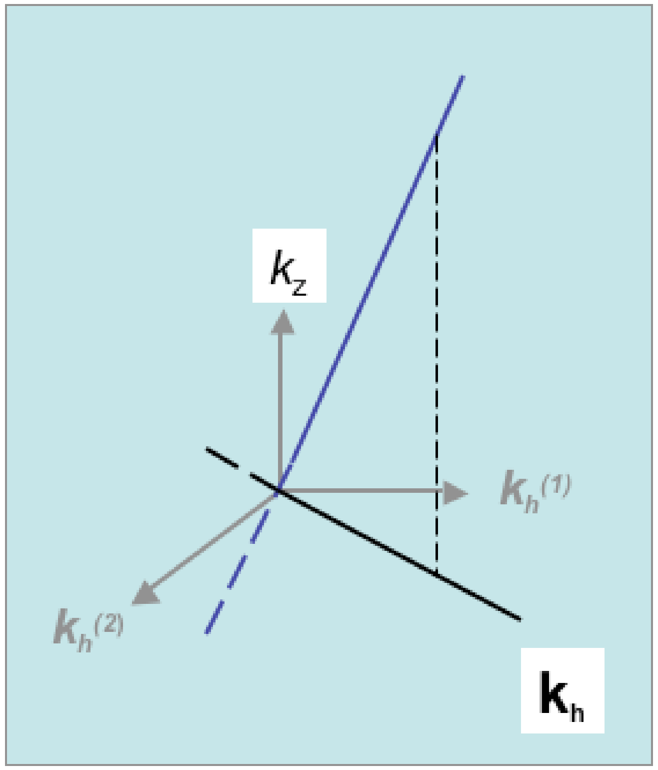

Figure 3: Each admissible triplet $(k_h^{(1)}, k_h^{(2)}, k_z)$ lies in a straight line of the Fourier domain

Angle-domain, common-image gathers construction

Apply the depth imaging condition, one for each offset $\mathbf{h}$, to obtain a collection of *offset gathers* $I(\mathbf{x}, z, \mathbf{h})$

Map offset gathers in depth by picking the proper pair $(k_h^{(1)}, k_h^{(2)})$, one for each k_z, to construct ADCI-gathers $J_{\alpha,\gamma,\varphi}(\mathbf{x}, z, \theta)$

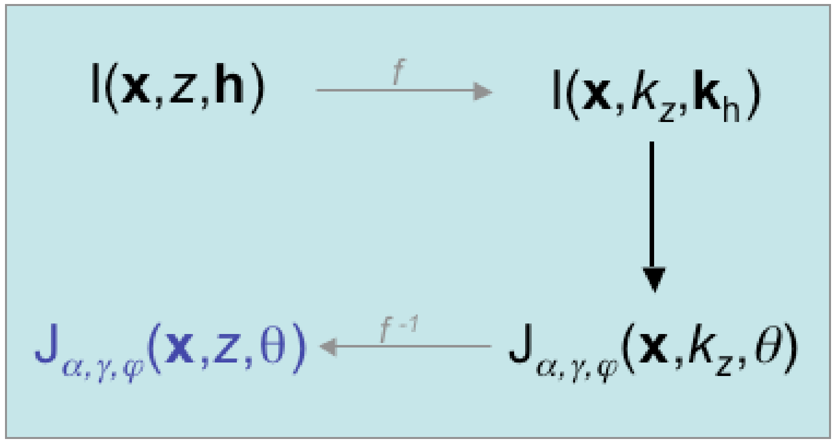

Figure 4: Imaging of all seismic events traveling on scattering planes defined by α, γ and φ, with reflection angle θ

Angle-domain, common-image gathers in 2D and depth imaging

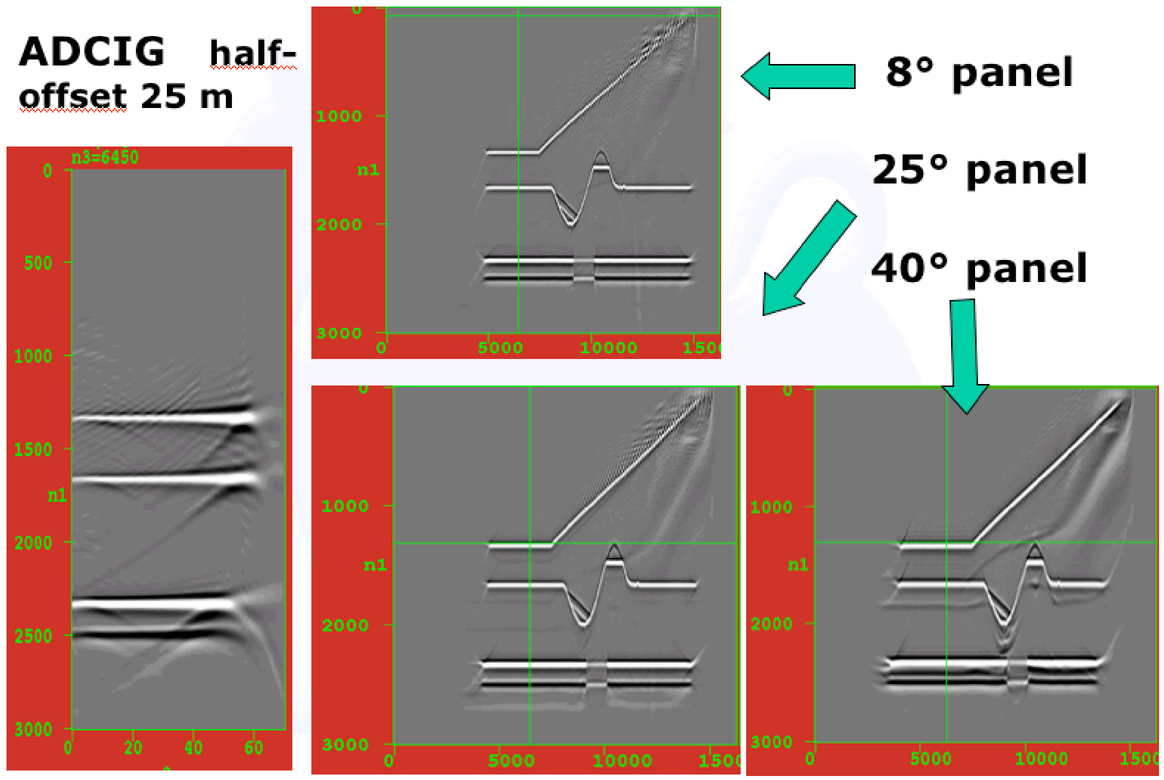

Figure 5: Depth imaging for three values of θ; on the left, the *depth-angle* panel displays the *alignment* of all events along the green vertical line (Luca Cazzola, ENI).

From the ADCI-gathers, back to the 2D medium image $(h = 0)$

In the Fourier domain, using $k_h = -\, k_z \tan\theta$, the ADCI panel takes the form:

$$J(x, k_z, \theta) = \int\int dz\ dh\ I(x, z, h)\ e^{ik_z(h\tan\theta - z)}\ .$$

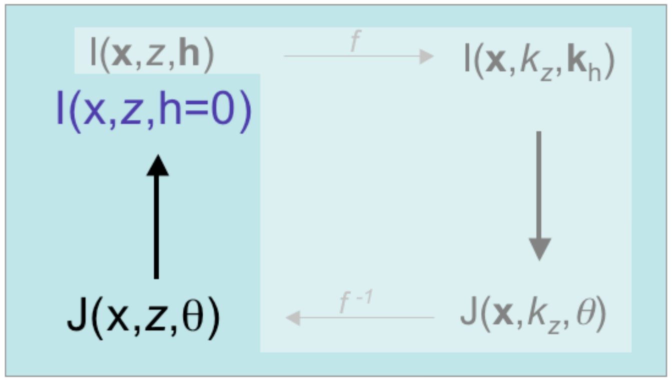

Figure 6: Mapping the *angle* domain panel onto the depth migrated image.

From the ADCI panel $J(x, z, \theta)$ we may reconstruct the **medium image** in the (x, k_z)-domain, implementing the following *summation* over all scattering angles θ:

$$\boxed{I(x, k_z, 0) = |k_z| \int_{-\pi/2}^{\pi/2} d\theta\ \frac{J(x, k_z, \theta)}{\cos^2\theta}\ .}$$