A Note on
The Reflection and Transmission Law of an Incident Wave

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I present here the classical boundary value problem of obliquely incident compressional plane waves impinging the interface between two homogeneous acoustic media. Each incident wave gives rise to reflected and transmitted components satisfying Snell's law. The reflection is usually quantified by the *reflection coefficient*, simply defined, on each point of the interface, as the ratio of the reflected and incident wave amplitudes.

Reflection coefficient: the boundary value problem

- Consider two homogeneous acoustic media having equal density but different propagation velocities v_I and v_T, separated by a plane boundary, Fig.1;
- An incident, *compressional* plane wave ϕ_I propagating across the first medium is partially reflected at the boundary and partially transmitted to the second medium:

$$\phi_I(\mathbf{x}, t) = \Re e \left\{ A \; e^{\imath\omega(t - \mathbf{p}_I \cdot \mathbf{x})} \right\}; \tag{1}$$

- Let us denote by $\mathbf{p}_I$ and $\mathbf{p}_R$ the slowness vectors characterizing, respectively, the incident wave orientation and the reflection direction:

$$\mathbf{p}_I = n_I \left(\cos\theta_I \; \mathbf{d} + \sin\theta_I \; \mathbf{d}_\perp \right), \qquad \mathbf{d} \cdot \mathbf{d}_\perp = 0, \tag{2}$$

$$\mathbf{p}_R = n_I \left(-\cos\theta_R \; \mathbf{d} + \sin\theta_R \; \mathbf{d}_\perp \right), \quad ||\mathbf{d}|| = ||\mathbf{d}_\perp|| = 1. \tag{3}$$

  $\mathbf{d}_\perp$ is a vector lying on the interface and orthogonal to $\mathbf{d}$; together, $\mathbf{d}$ and $\mathbf{d}_\perp$ define the *reflection plane*; $n_I = 1/v_I$ is the slowness value of the first medium;
- Let us characterize the transmission direction in the second medium by the slowness vector $\mathbf{p}_T$: $||\mathbf{p}_T|| = n_T$, where $n_T = 1/v_T$.

Medium 1: The assumed form of the pressure wavefield is at each time t, on each point $\mathbf{x}$ of the medium, $\Phi_1 = \phi_I + \phi_R$, where ϕ_R indicates the reflected component of the incident plane wave:

$$\phi_R(\mathbf{x}, t) = \Re e \left\{ B \; e^{\imath\omega(t - \mathbf{p}_R \cdot \mathbf{x})} \right\}, \quad ||\mathbf{p}_R|| = n_I. \tag{4}$$

Medium 2: The assumed form of the pressure wavefield is at each time t, on each point $\mathbf{x}$ of the medium, $\Phi_2 = \phi_T$, where ϕ_T denotes the transmitted component of the incident plane wave:

$$\phi_T(\mathbf{x}, t) = \Re e \left\{ C \; e^{\imath\omega(t - \mathbf{p}_T \cdot \mathbf{x})} \right\}, \quad ||\mathbf{p}_T|| = n_T. \tag{5}$$

- To complete the description of the plane wave scattering in terms of θ_I, v_I, and v_T, some condition of *continuity* must be provided to patch Φ_1 and Φ_2 across the interface separating the two media;
- The patching condition must lead to four relations for the evaluation of $R = \frac{B}{A}$ and $T = \frac{C}{A}$, and the reflection and transmission angles θ_R and θ_T;
- Because the two media are homogeneous, R and T can only be *real* numbers; they are called, respectively, the *reflection* and the *transmission* coefficients.

Conditions of continuity across the interface

- Without loss of generality we can assume that the plane interface contains the origin of the global reference system.

Condition 1: We impose for every time t the continuity of the pressure field on each point $\mathbf{x}_0$ of the boundary, $\Phi_1 = \Phi_2$:

$$e^{-\imath\omega\mathbf{p}_I\cdot\mathbf{x}_0} + R\ e^{-\imath\omega\mathbf{p}_R\cdot\mathbf{x}_0} = T\ e^{-\imath\omega\mathbf{p}_T\cdot\mathbf{x}_0}. \tag{6}$$

Condition 2: We impose for every time t the continuity of the first spatial derivatives of the wavefield on each point $\mathbf{x}_0$ of the boundary, $\frac{\partial}{\partial x_i}\Phi_1 = \frac{\partial}{\partial x_i}\Phi_2$:

$$\mathbf{p}_I\ e^{-\imath\omega\mathbf{p}_I\cdot\mathbf{x}_0} + \mathbf{p}_R\ R\ e^{-\imath\omega\mathbf{p}_R\cdot\mathbf{x}_0} = \mathbf{p}_T\ T\ e^{-\imath\omega\mathbf{p}_T\cdot\mathbf{x}_0}. \tag{7}$$

- Expression (7) sets $\mathbf{p}_T$ as a linear combination of $\mathbf{p}_I$ and $\mathbf{p}_R$, meaning that even the transmitted slowness vector *lies* on the reflecting plane spanned by $\mathbf{d}$ and $\mathbf{d}_\perp$:

$$\mathbf{p}_T = n_T\left(\cos\theta_T\ \mathbf{d} + \sin\theta_T\ \mathbf{d}_\perp\right). \tag{8}$$

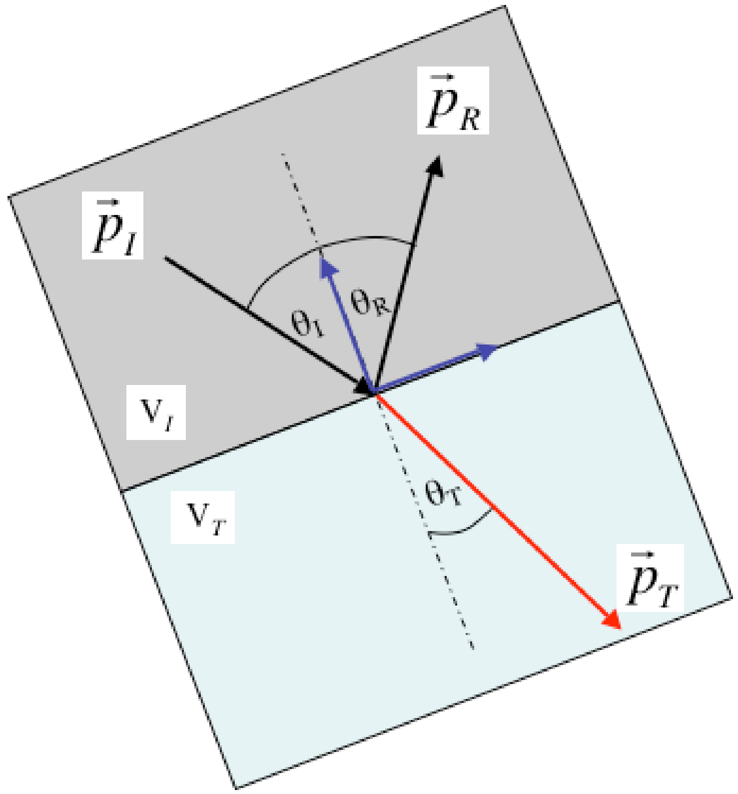

Figure 1: Schematic representation of incident, reflected, and transmitted waves.

Snell's law at the interface

- Projecting twice the vector equation (8), respectively onto $\mathbf{d}$ and $\mathbf{d}_\perp$, we find - using (2), (3) and (8) - the following two scalar relations,

$$n_I \cos\theta_I \; e^{-\imath\omega \mathbf{p}_I\cdot\mathbf{x}_0} - n_I \cos\theta_R \; R \; e^{-\imath\omega \mathbf{p}_R\cdot\mathbf{x}_0} = n_T \cos\theta_T \; T \; e^{-\imath\omega \mathbf{p}_T\cdot\mathbf{x}_0}, \tag{9}$$

$$n_I \sin\theta_I \; e^{-\imath\omega \mathbf{p}_I\cdot\mathbf{x}_0} + n_I \sin\theta_R \; R \; e^{-\imath\omega \mathbf{p}_R\cdot\mathbf{x}_0} = n_T \sin\theta_T \; T \; e^{-\imath\omega \mathbf{p}_T\cdot\mathbf{x}_0}, \tag{10}$$

  that, together with equation (6), $e^{-\imath\omega \mathbf{p}_I\cdot\mathbf{x}_0} + R \; e^{-\imath\omega \mathbf{p}_R\cdot\mathbf{x}_0} = T \; e^{-\imath\omega \mathbf{p}_T\cdot\mathbf{x}_0}$, form an over-determined linear system in R and T.

Hypothesis 1: By setting

$$n_I \sin\theta_I = n_I \sin\theta_R = n_T \sin\theta_T, \tag{11}$$

equation (10) is transformed into (6) and, as a matter of fact, the initial linear system may now recover the proper rank to exhibit a unique solution.

- Condition (11), illustrated in Fig.2, is well known as *Snell's law*: it relates incident, reflection and transmission directions of the wavefield to the propagation velocity in each one of the two adjacent media.

- Thanks to Snell's law, the reduced linear system may now be written as follows:

$$\begin{array}{rcrcr} n_I \cos\theta_I \; e^{-\imath\omega \cos\theta_I \mathbf{d}\cdot\mathbf{x}_0} & = & n_I \cos\theta_R \; R \; e^{-\imath\omega \cos\theta_R \mathbf{d}\cdot\mathbf{x}_0} & + & n_T \cos\theta_T \; T \; e^{-\imath\omega \cos\theta_T \mathbf{d}\cdot\mathbf{x}_0}, \\ -e^{-\imath\omega \cos\theta_I \mathbf{d}\cdot\mathbf{x}_0} & = & R \; e^{-\imath\omega \cos\theta_R \mathbf{d}\cdot\mathbf{x}_0} & - & T \; e^{-\imath\omega \cos\theta_T \mathbf{d}\cdot\mathbf{x}_0}. \end{array} \tag{12}$$

Hypothesis 2: Because R and T must be *real* solutions, we impose on (12) to have real coefficients by requiring $\mathbf{d} \cdot \mathbf{x}_0 = 0$ for every point $\mathbf{x}_0$ on the interface:

$$\begin{array}{rcrcr} n_I \cos\theta_I & = & n_I \cos\theta_R \; R & + & n_T \cos\theta_T \; T, \\ -1 & = & R & - & T. \end{array} \tag{13}$$

With this requirement we have imposed the reflection plane - spanned by $\mathbf{d}$ and $\mathbf{d}_\perp$ - to be *orthogonal* to the interface between the two media.

- Finally, from (13) we can estimate the reflection and the transmission coefficients:

$$R = \frac{n_I \cos\theta_I - n_T \cos\theta_T}{n_I \cos\theta_I + n_T \cos\theta_T} \; , \qquad T = \frac{2 \; n_I \cos\theta_I}{n_I \cos\theta_I + n_T \cos\theta_T} \; . \tag{14}$$

These two relations must, however, be completed with the companion equation (11).

- Combining (14) and (11), we obtain two expressions only depending on the incident angle θ_I and the ratio of v_I to v_T, here denoted as z:

$$R = \frac{z - \sqrt{1 + \tan^2\theta_I\,(1 - z^2)}}{z + \sqrt{1 + \tan^2\theta_I\,(1 - z^2)}}\ , \quad T = \frac{2z}{z + \sqrt{1 + \tan^2\theta_I\,(1 - z^2)}}, \quad z = \frac{v_T}{v_I}\ . \quad (15)$$

  Remark 1: Because R end T must be real numbers, to fulfill this requirement we need $1 + \tan^2\theta_I\,(1 - z^2) \geq 0$, a condition constraining θ_I to z:
  - if $z \leq 1$, then $0 \leq |\theta_I| \leq \pi/2$;
  - if $z > 1$, then $0 \leq |\theta_I| \leq \arctan(\frac{1}{\sqrt{z^2-1}})$.

- Transmission and reflection of the incident wavefront, see Fig.(2), are then controlled by the contrast of velocity of one medium with regard to the other.

  Remark 2: As long as two adjacent media are homogeneous, R and T do *not* depend on the deep angle of the reflecting interface.

- Under the condition of homogeneity, both expressions (15) remain valid in the case of non planar interfaces.

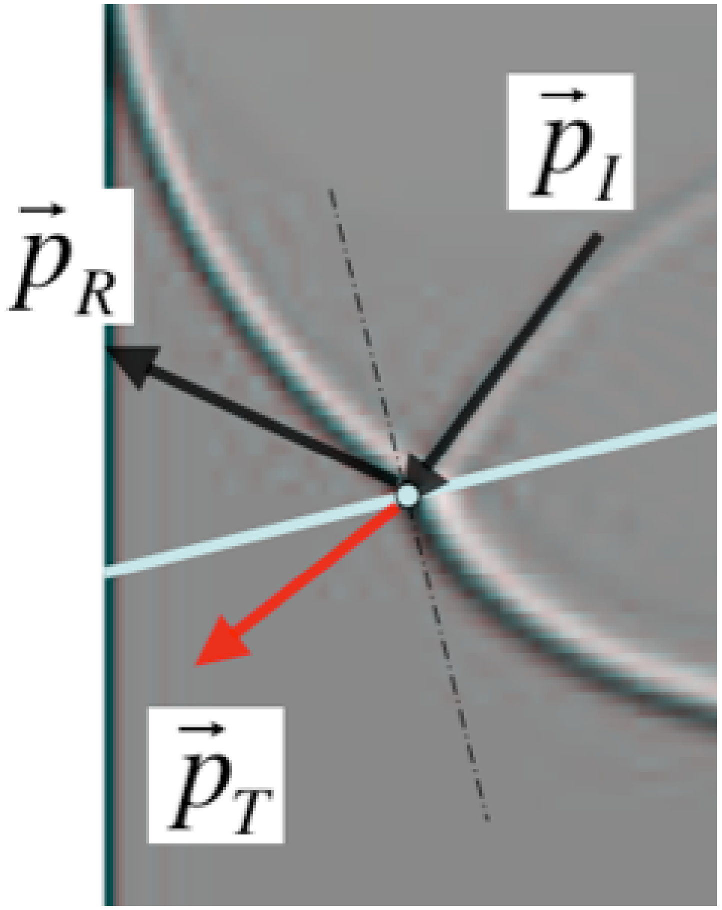

Figure 2: Spherical incident wavefront at the contact point with an interface, $z > 1$, and the resulting reflected and transmitted components: the three slowness vectors are normal to the propagating fronts and, together, they satisfy *Snell's law*.