CRS4 internal report: Square Additive Approach to Turbulent Prandtl Number

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1 Introduction

Low Prandtl number liquid metals serve as primary coolant for MYRRHA and ALFRED, two Gen-IV reactors under development [1]. The low Prandtl number induces discrepancies in the modeling of the turbulent heat transfer when directly treated according to the Reynolds analogy with a constant turbulent Prandtl number. The thermal boundary layer is considerably larger than the velocity one, up to the point that, while we can clearly define a bulk velocity, there can be no such thing as a bulk temperature. Many correlations have been derived to tackle the issue [2, 3, 4]. Almost all of them use global parameters such as the Reynolds or Peclet numbers which are not well defined in complex geometries and by such not suitable for the related CFD simulations. To the author knowledge, only Kays' correlation is using only local parameters. Several variants of the correlations have been used with significant success, leading to some perplexity. In a former paper [5], we showed that the correlations can be simply derived on a basic assumption with regards to the non-linear combination of stochastic effects and the variants then come from different approximations of a mother formula. A defect of the mother formula, which is transferred to the variants is that the turbulent Prandtl number becomes infinite at vanishing turbulence. In this paper, we proceed further with the constructive hypothesis to refine the formula, extending the principle of square additivity not only to the thermal conductivity but also to the viscosity. The derived formula has the merit to be simple and to not degenerate any more at vanishing turbulence.

2 Derivation of Pr_t and its approximation

In CFD codes, both viscous effects and heat diffusion modelling and implementation are represented as the direct sum of two contributions, the molecular one and the turbulent one. With regard to the heat diffusion, both contributions act in the same direction, proportionally to the local temperature gradient. Each contribution is also proportional to a conductivity coefficient. In other words

the effective conductivity coefficient k_e is given as

$$k_e = k + k_t \tag{1}$$

where k is the molecular conductivity coefficient observed in static or laminar flows and k_t is the increment due to turbulence. With ρ the fluid density and C_p its specific heat, this can be rewritten in terms of thermal diffusivity:

$$\alpha_e = \alpha + \alpha_t \tag{2}$$

in which $\alpha_e = k_e/(\rho C_p)$ is the effective diffusivity, $\alpha = k/(\rho C_p)$ is the molecular part and $\alpha_t = k_t/(\rho C_p)$ is the turbulent part. Similarly, with regards to the viscosity the effective kinematic viscosity ν_e, is the sum of a laminar contribution ν and a turbulent contribution ν_t.

$$\nu_e = \nu + \nu_t \tag{3}$$

The argument developed in precedence [5] is the following. Conduction in general is originated from a stochastic process. Molecular conduction and turbulent conduction look like different unrelated mechanisms operating at different scales. While molecular conduction is in fact a molecular process, turbulent conduction is rather based on a convective process. The cumulative effect is a convolution rather than a mere juxtaposition. In turn, one would expect the combined effects to be according to:

$$\alpha_e = \sqrt{\alpha^2 + \alpha_0^2}, \tag{4}$$

where α_0 is the turbulent thermal diffusivity expected for a similar fluid with (almost) zero molecular conductivity or similarly when the turbulence is extremely large.

We now extend the argument to the viscous process and state that the effective viscosity comes from two independent processes whose intensity should be square additive.

$$\nu_e = \sqrt{\nu^2 + \nu_0^2}. \tag{5}$$

By simple substitution, we have just defined two quantities:

$$\nu_0 = \nu_t \sqrt{1 + \frac{2\nu}{\nu_t}}. \tag{6}$$

and

$$\alpha_0 = \alpha_t \sqrt{1 + \frac{2\alpha}{\alpha_t}}. \tag{7}$$

We can redefine ν_t and α_t in terms of these quantities:

$$\nu_t = \sqrt{\nu^2 + \nu_0^2} - \nu \tag{8}$$

$$\alpha_t = \sqrt{\alpha^2 + \alpha_0^2} - \alpha \tag{9}$$

The (molecular) Prandlt numbers is defined as $Pr = \frac{\nu}{\alpha}$. Similarly, the turbulent Prandtl number is defined as $Pr_t = \frac{\nu_t}{\alpha_t}$ and the asymptotic turbulent Prandtl number as $Pr_0 = \frac{\nu_0}{\alpha_0}$. This later number has been conceived in order to be the most possible independent of the fluid. Noting that $\frac{\alpha}{\alpha_0} = \frac{Pr_0\nu}{Pr\nu_0}$, simple algebraic manipulations from the former equations give:

$$Pr_t = \frac{Pr_0^2}{Pr} \frac{\sqrt{1 + (\frac{Pr\nu_0}{Pr_0\nu})^2} + 1}{\sqrt{1 + (\frac{\nu_0}{\nu})^2} + 1}. \tag{10}$$

This specific form is chosen to show that is is "difficult" to make degenerate to infinity. In effect, because ν_0 goes to zero if ν_t does, then Pr_t tend to $\frac{Pr_0^2}{Pr}$ while forge large turbulence, Pr_t tends to Pr_0. In turn, if Pr_0, built for this purpose, do not degenerate, then neither do Pr_t.

This formulation is practical only if ν_0 is readily available. This would be the case if we had a transport equation for ν_0 or a related variable similarly to what is done with the usual 2-equations turbulence models. Exploring the potential of this possibility is beyond the scope of the current argument and we need to express Pr_t in terms of known parameters.

Thus, expressing Pr_t, in terms of ν_t instead of ν_0, the formula reduces to:

$$Pr_t = \frac{Pr_0^2}{Pr} \frac{(2 + \frac{Pr\nu_t}{Pr_0\nu})}{2 + \frac{\nu_t}{\nu}}. \tag{11}$$

this form being useful for interpretation at vanishing turbulent viscosity. Or

$$Pr_t = Pr_0 \frac{(1 + \frac{2Pr_0\nu}{Pr\nu_t})}{1 + \frac{2\nu}{\nu_t}} \tag{12}$$

better suited for larger ν_t. In particular, when $\nu_t > 20\nu$, the former expression can be approximated within $\sim 1\%$ under a Kays correlation [2] like form:

$$Pr_t \simeq Pr_0 + 2\frac{Pr_0(Pr_0 - Pr)}{\frac{Pr\nu_t}{\nu}} \tag{13}$$

Taking into consideration that Pr_0 is usually taken as $Pr_0 = 0.85$, we get for $Pr \simeq 0.02$:

$$Pr_t = 0.85 + \frac{1.41}{Pr\frac{\nu_t}{\nu}} \tag{14}$$

For the second coefficient (here 1.41), Kays indicated two values, 0.7 and 2 discussing without reaching a conclusion in favor of one or the other value. However, the current value of 1.41 is quite close to the value value of 1.46 used by [6] to better fit heat transfer in a tube by direct analytical integration.

There are indication [2] that Pr_t and therefore also Pr_0 is about 0.85 for very large ν_t/ν. We do not know the behavior of Pr_0 for lower values. Nevertheless, we expect a rather constant behavior down to (close to) the Kolmogorov scale when the laminar viscosity dominates the turbulent one. In this later case, the

hypothesis of independence of the stochastic processes becomes questionable. It is also interesting to observe that for medium and high Pr, then Pr_t in formula (12) to not significantly departs from Pr_0.

While the derived formula is thought to be used within a turbulence model it can be so only if the turbulence model correctly predicts the turbulent viscosity, not only in the viscous boundary layer but also and principally in the bulk. The problem is that the turbulence models mainly focus on a correct boundary layer turbulent viscosity profile, as it is the place where almost all the pressure drop is built. The turbulent viscosity profile in the bulk is normally of no practical importance, except for thermal flows of low Prandtl fluids.

3 Conclusion

We consider that turbulent and molecular viscosity are originated from two independent stochastic processes whose intensity is square additive. We make the same consideration for the turbulent and the thermal conductivity. As a direct consequence, comes a simple expression for the Prandtl number which first order approximation is very similar to a variant of Kays' correlation, the only non-trivial correlation exclusively based on local parameters and thus suited for arbitrary CFD simulations. The formula introduces and depends on only one parameter, the asymptotic Prandtl number which is built to be the less possible dependent on the fluid physical properties and is basically known at large turbulence.

References

[1] M. Tarantino, F. Roelofs, A. Shams, A. Batta, Vincent Moreau, I. Di Piazza, A. Gershenfeld, P. Planquart "SESAME project: advancements in liquid metal thermal hydraulics experiments and simulations" EPJ Nuclear Sci. Technol., Volume 6, Number 18 - july 2020

[2] W. M. Kays, "Turbulence Prandtl Number - Where are we?", Journal of Heat transfer, Vol.16/285 May 1994

[3] Xu Cheng, Nam-il Tak "Investigation on turbulent heat transfer to lead-bismuth eutectic flows in circular tubes for nuclear applications", Nucl. Eng. Des. 236 (2006) 385-393.

[4] Y. Bartosiewiccz, M. Duponcheel, M. Mancono, G. Winckelmans and L. Bricteux, "Turbulence Modeling at Low Prandtl Number" Chapter in "'Fluid Mechanics and Chemistry for Safety Issuest in HLM Nuclear Reactors" Lecture Series 2014-02 by the von Karman Institute for Fluid Dynamics, ISBN-13 978-2-87516-064-5

[5] V. Moreau "Turbulent Prandtl Number, reformulation of Kay's correlation" Academia Letters DOI: 10.20935/AL2366 Volume: Article 2366 year 2021

[6] Dawid Taler, "Heat transfer in turbulent tube flow of liquid metals", Procedia Engineering 157 (2016) 148 – 157, IX International Conference on Computational Heat and Mass Transfer, ICCHMT2016 doi: 10.1016/j.proeng.2016.08.350

[7] Vodret S, Vitale Di Maio D and Caruso G 2014 J. Phys.: Conf. Ser. 547 012033

[8] F Vitillo, D Vitale Di Maio, C Galati, G Caruso, "An anisotropic numerical model for thermal hydraulic analyses: application to liquid metal flow in fuel assemblies" 3rd UIT (Italian Union of Thermo-fluid-dynamics) Heat Transfer ConferenceIOP Publishing Journal of Physics: Conference Series 655(2015) 012058 doi:10.1088/1742-6596/655/1/012058